A Dynamic Model of Circuit Breakers

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December 5, 2016
Preliminary and Incomplete

Abstract

We build a dynamic model to examine the mechanism through which market-wide circuit breakers affect trading and price dynamics in the stock market. We show that the presence of downside circuit breakers tends to lower the price-dividend ratio, reduce daily price ranges, but increase conditional and realized volatilities. They also raise the probability of the stock price reaching the circuit breaker limit as the price approaches the threshold (a "magnet" effect). The effects of circuit breakers can be further amplified when some agents' willingness to hold the stock is sensitive to recent shocks to fundamentals, which can be due to behavioral biases, institutional constraints, etc. Surprisingly, the volatility amplification effect of circuit breakers is the strongest when the initial wealth share for the irrational agent is the smallest. Finally, using historical data from a period when circuit breakers have not been implemented can lead one to underestimate the likelihood of triggering a circuit breaker, especially when the threshold is relatively tight.

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1 Introduction

In this paper, we present a dynamic asset pricing model to examine the mechanism through which market-wide circuit breakers affect trading and price dynamics in the stock market.

Circuit breakers are procedures that halt trading temporarily or close the market for the remainder of the trading day when the market price of a security or an index moves by a significant amount. For example, in the U.S. equity markets, a cross-market trading halt can be triggered at three circuit breaker thresholds: 7% (Level 1), 13% (Level 2), both of which will halt market-wide trading for 15 minutes when the decline occurs between 9:30 a.m. and 3:25 p.m. Eastern time, and 20% (Level 3), which halts market-wide trading for the remainder of the trading day. These triggers are set based on the prior day's closing price of the S&P 500 Index. Circuit breakers were first introduced by the Brady Commission following the Black Monday of 1987, and they have been widely adopted by equity and derivative exchanges around the world. Besides downside circuit breakers as those in the U.S., some markets also have upside circuit breakers.

The China Securities Regulatory Commission (CSRC) started implementing a market-wide circuit breaker in January 2016, with a 15-minute trading halt when the CSI 300 Index falls by 5% (Level 1) from previous day's close, and market closure after a 7% decline (Level 2). On January 4, 2016, the first trading day after the rule was put in place, both thresholds were reached (Figure 1, left panel), and it took only 7 minutes from the re-opening of the markets following the 15-minute halt for the index to reach the 7% threshold. Three days later, on January 7, both circuit breakers were triggered again (Figure 1, right panel), and the entire trading session lasted just 30 minutes. On the same day, the CSRC suspended the circuit breaker rule.

These events have revived debates among academics and practitioners about circuit breakers. Did the circuit breakers cause or amplify the declines in the Chinese stock market? Are market-wide circuit breakers needed for markets like in China, where

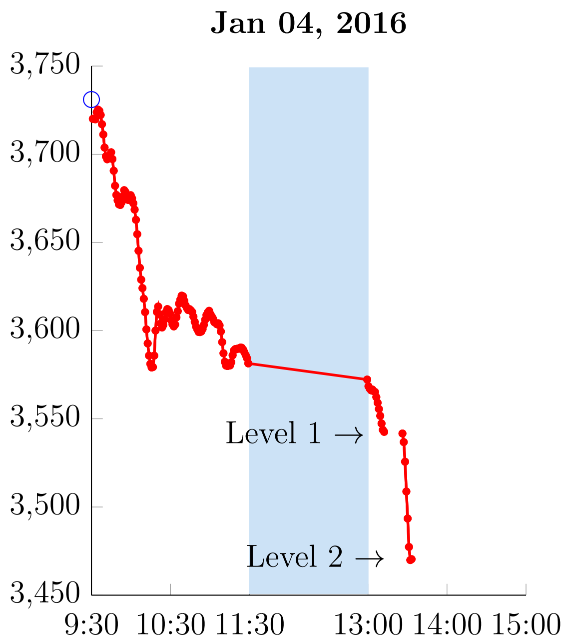

Figure 1: **Circuit breakers in the Chinese stock market.** The two panels plot the CSI300 index on January 4 and January 7 of 2016, when the circuit breaker was triggered. Trading hours for the Chinese stock market are 9:30-11:30 and 13:00-15:00. Level 1 (2) circuit breaker is triggered after a 5% (7%) drop in price from the previous day's close (marked by the blue circles).

price limits for individual stocks are already in place? Should they be designed differently from those in other more developed economies (including the level of the threshold and the length of the trading halts)?

Among the commonly stated rationales for circuit breakers are to reduce excess volatility, to restore orderly trading, and to restore confidence among investors. Greenwald and Stein (1991) argue that, in the presence of informational frictions, trading halts can help improve the information available to market participants and in turn the efficiency of allocations. On the other hand, Subrahmanyam (1994) argue that circuit breakers can increase price volatility by causing investors (especially institutional investors) to suboptimally advance their trades in time, for fear that that a halt might occur before they can submit their orders. Furthermore, building on the insights of Diamond and Dybvig (1983), Bernardo and Welch (2004) show that, when facing the threat of future liquidity shocks, coordination failures can lead to runs and high volatility in the financial markets. Such mechanisms could also increase price volatility in the presence of circuit breakers.

Different from the papers above, we study the impact of circuit breakers in an otherwise frictionless model, abstracting away from trading frictions, information asymmetry, and coordination failures. By doing so, we provide a new benchmark to demonstrate the impact of potential trading halts on trading and price dynamics. Moreover, while the above mentioned papers focus on qualitative predictions, our model is dynamic and can be calibrated to study the quantitative impacts of circuit breakers. This feature is crucial for studying how different designs of circuit breaker rules affect the financial markets.

In our model, two investors have log preferences over terminal wealth and have heterogeneous beliefs about the dividend growth rate. Without circuit breakers, the stock price is a weighted average of the prices under the two agents' beliefs, with the weights being their respective shares of total wealth.

However, the presence of circuit breakers makes the equilibrium stock price disproportionately reflect the beliefs of the relatively pessimistic agent. To understand this, first consider the scenario when the stock price has just reached the circuit breaker threshold. Immediate market closure is an extreme form of illiquidity, which forces the relatively optimistic investors to refrain from taking on any leverage due to the inability to rebalance his portfolio. As a result, the pessimistic agents the marginal investors, and the equilibrium stock price has to entirely reflect his beliefs, regardless of his wealth share.

The threat of market closure also affects trading and prices before the circuit breaker is triggered. Compared to the case without circuit breakers, the relatively optimistic investor will preemptively reduce his leverage as the price approaches the circuit breaker limit. In the case of a downside circuit breaker, the price-dividend ratios are driven lower, while the conditional volatilities of stock returns can become significantly higher. In the case of an upside circuit breaker, the price-dividend ratios are again lower than without circuit breakers, but return volatility will also become lower. These effects are stronger when the price is closer to the circuit breaker threshold, when it is earlier during a trading session. Surprisingly, the volatility amplification effect of downside circuit breakers is the strongest when the initial wealth share for the irrational (pessimistic) agent is the smallest.

Our model shows that circuit breakers have multifaceted effects on price volatility. On the one hand, a (tighter) downside circuit breaker limit can lower the median daily price range (measured by daily high minus low prices) and reduce the probabilities of very large daily price ranges. Such effects could be beneficial, for example, in reducing inefficient liquidations due to intra-day mark-to-market. On the other hand, a (tighter) downside circuit breaker will tend to raise the probabilities of intermediate price ranges, and can significantly increase the median of daily realized volatilities as well as the probabilities of very large conditional and realized volatilities. These effects could exacerbate market instability.

Furthermore, our model demonstrates a "magnet effect." The very presence of downside circuit breakers makes it more likely for the stock price to reach the threshold in a given amount of time than when there are no circuit breakers (the opposite is true for upside circuit breakers). The difference between the probabilities is negligible when the stock price is sufficiently far away from the threshold, but it generally gets bigger as the stock price gets closer to the threshold. Eventually, when the price is sufficiently close to the threshold, the gap converges to zero as both probabilities converge to one.

This "magnet effect" is important for the design of circuit breakers. It suggests that using the historical data from a period when circuit breakers were not implemented can lead one to severely underestimate the likelihood of future circuit breaker triggers, which might result in picking a downside circuit breaker limit that is excessively tight.

We also examine the case with one of the agents having time-varying beliefs that are positively correlated with the dividend shocks. Such beliefs may represent the "representativeness" bias in behavioral economics, or they could represent constrained investors who behave as if they are becoming more pessimistic (risk averse) when the negative dividend shocks result in losses that tighten their constraints. In this case, circuit breakers can have even stronger effects on lowering the price-dividend ratio and amplifying the conditional return volatility near the circuit breaker threshold.

2 The Model

We consider a continuous-time endowment economy over the finite time interval $[0, T]$. Uncertainty is described by a one-dimensional standard Brownian motion Z, defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the augmented filtration generated by Z.

There is a single share of an aggregate stock, which makes a single terminal dividend payment of D_T at time T. The process for D is exogenous and publicly observable,

$$dD_t = \mu D_t dt + \sigma D_t dZ_t, \quad D_0 = 1, \tag{1}$$

where μ and $\sigma > 0$ are the expected growth rate and volatility of D_t.[1] Besides the aggregate stock, there is also a riskless bond which is in zero net supply. Each unit of the bond pays off one at time T.

There are two competitive agents A and B, who are initially endowed with θ and $1 - \theta$ shares of the aggregate stock, with $0 \leq \theta \leq 1$. Both agents have logarithmic preferences over their terminal wealth at time T,

$$u_i(W_T^i) = \ln(W_T^i), \quad i = \{A, B\}.$$

There is no intermediate consumption.

The two agents have heterogeneous beliefs about the value of the terminal dividend.

[1]For brevity, throughout the paper we will refer to D_t as "dividend" and S_t/D_t as the "price-dividend ratio," even though dividend will only be realized at time T.

Agent A has objective beliefs in the sense that his probability measure is consistent with $\mathbb{P}$ (in particular, $\mu^A = \mu$). Agent B's belief about the growth rate is

$$\mu_t^B = \mu + \delta_t, \tag{2}$$

where the difference in belief δ_t follows an Ornstein-Uhlenbeck process,

$$d\delta_t = \kappa(\bar{\delta} - \delta_t)dt + \nu dZ_t. \tag{3}$$

Notice that δ_t is driven by the same Brownian motion as the aggregate dividend. With $\nu > 0$ and $\kappa > 0$, agent B becomes more optimistic following positive shocks to the aggregate dividend, and the impact of these shocks on his belief decays exponentially at the rate κ. Thus, the parameter ν controls how sensitive B's conditional belief is to realized dividend shocks, while κ determines the relative importance of shocks from recent past vs. distant past. The average long-run disagreement between the two agents is $\bar{\delta}$. In the special case with $\nu = 0$ and $\delta_0 = \bar{\delta}$, we get constant disagreement between the two agents.

Interpretation: time-varying beliefs could represent behavioral biases ("representativeness") or a form of path-dependent utility that makes agent B more (less) risk averse following negative (positive) shocks to fundamentals.

Agent B's probability measure is $\mathbb{P}^B$, which we shall suppose is equivalent to $\mathbb{P}$.[2] We assume that the two agents are aware of each others' beliefs but "agree to disagree."[3] Let the Radon-Nikodym derivative of the probability measure $\mathbb{P}^B$ with respect to $\mathbb{P}$ be η. Then from Girsanov's theorem, we get

$$\eta_t = \exp\left(\frac{1}{\sigma}\int_0^t \delta_s dZ_s - \frac{1}{2\sigma^2}\int_0^t \delta_s^2 ds\right). \tag{4}$$

Intuitively, since agent B will be more optimistic than A when $\delta_t > 0$, those paths

[2] More precisely, $\mathbb{P}$ and $\mathbb{P}^B$ are equivalent when restricted to any σ-field $\mathcal{F}_T = \sigma(\{D_t\}_{0\le t\le T})$.
[3] We do not explicitly model learning ...

with high realized values for $\int_0^t \delta_s Z_s$ will be assigned higher probabilities under $\mathbb{P}^B$ than under $\mathbb{P}$.

Because there is no intermediate consumption, we use the riskless bond as the numeraire.

Circuit Breakers. To capture the essence of a circuit breaker rule, we assume that the stock market will be closed whenever the price of the stock S_t falls below a threshold $(1-\alpha)S_0$, where S_0 is the endogenous initial price of the stock, and $\alpha \in [0, 1]$ is a constant parameter determining the bandwidth of downside price fluctuations during the interval $[0, T]$. Later in Section **??**, we extend the model to allow for market closure for both downside and upside price movements, which represent price limit rules. The closing price for the stock is determined such that both the stock market and bond market are cleared when the circuit breaker is triggered. After that, the stock market will remain closed until time T. The bond market remains open throughout the interval $[0, T]$.

In practice, the circuit breaker threshold is often based on the closing price from the previous trading session instead of the opening price of the current trading session. For example, in the U.S., a cross-market trading halt can be triggered at three circuit breaker thresholds (7%, 13%, and 20%) based on the prior days closing price of the S&P 500 Index. However, the distinction between today's opening price and the prior day's closing price is not crucial for our model. The circuit breaker not only depends on but also endogenously affects the initial stock price, just like it does for prior day's closing price in practice.[4]

Finally, we impose usual restrictions on trading strategies to rule out arbitrage.

[4]Other realistic features of the circuit breaker in practice: closing the market for x minutes and reopen (Level 1 and 2), or closing the market until the end of the day (Level 3). Analogy to our model: Think of T as one day. The fact that the price of the stock reverts back to the fundamental value X_T at T resembles the rationale of CB to "restore order" in the market.

3 The Equilibrium

3.1 Benchmark: No Circuit Breakers

In this section, we solve for the equilibrium when there are no circuit breakers. To distinguish the notations from the case with circuit breakers, we use the symbol "$\widehat{\ }$" to denote variables in the case without circuit breakers.

In the absence of circuit breakers, markets are dynamically complete. The equilibrium allocation in this case can be characterized as the solution of the following planner's problem,

$$\max_{\widehat{W}_T^A, \widehat{W}_T^B} \mathbb{E}_0 \left[\lambda \ln \left(\widehat{W}_T^A \right) + (1-\lambda)\eta_T \ln \left(\widehat{W}_T^B \right) \right], \tag{5}$$

subject to the resource constraint

$$\widehat{W}_T^A + \widehat{W}_T^B = D_T. \tag{6}$$

From the first-order conditions and the resource constraint we then get

$$\widehat{W}_T^A = \frac{\lambda}{\lambda + (1-\lambda)\eta_T} D_T, \tag{7a}$$

$$\widehat{W}_T^B = \frac{(1-\lambda)\eta_T}{\lambda + (1-\lambda)\eta_T} D_T. \tag{7b}$$

Thus, as it follows from the intuition we have for the Radon-Nikodym derivative η_t, the optimistic agent B will be allocated a bigger share of the aggregate dividend under those paths with higher realized growthes in dividend.

The state price density under agent A's beliefs (i.e., the objective probability measure $\mathbb{P}$) is

$$\widehat{\pi}_t^A = \mathbb{E}_t[\xi u'(\widehat{W}_T^A)] = \mathbb{E}_t \left[\xi (\widehat{W}_T^A)^{-1} \right], \quad 0 \le t \le T \tag{8}$$

for some constant ξ. Then, from the budget constraint for agent A we see that the planner weights are equal to the shares of endowment, $\lambda = \theta$.

The following proposition summarizes the pricing results:

Proposition 1. *When there are no circuit breakers, the price of the aggregate stock is*

$$\widehat{S}_t = \frac{\theta + (1-\theta)\,\eta_t}{\theta + (1-\theta)\,\eta_t e^{a(t,T)+b(t,T)\delta_t}} D_t e^{\left(\mu-\sigma^2\right)(T-t)}, \tag{9}$$

where

$$\begin{aligned} a\,(t,T) &= \left(\frac{\kappa\bar{\delta}-\sigma\nu}{\frac{\nu}{\sigma}-\kappa} + \frac{\nu^2}{2\left(\frac{\nu}{\sigma}-\kappa\right)^2}\right)(T-t) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma}-\kappa\right)^3}\left(1-e^{2\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right) \\ &\quad + \left(\frac{\kappa\bar{\delta}-\sigma\nu}{\left(\frac{\nu}{\sigma}-\kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma}-\kappa\right)^3}\right)\left(1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right), && \text{(10a)} \\ b\,(t,T) &= \frac{1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}}{\frac{\nu}{\sigma}-\kappa}. && \text{(10b)} \end{aligned}$$

Proof. See Appendix A. □

From Equation (9), we can then derive the conditional volatility of the stock $\widehat{\sigma}_{S,t}$ in closed form, which we present in the appendix.

Next, we turn to the wealth distribution and portfolio holdings of individual agents. At time $t \leq T$, the shares of total wealth of the two agents are

$$\widehat{\phi}_t^A = \frac{\widehat{W}_t^A}{\widehat{S}_t} = \frac{\theta}{\theta + (1-\theta)\eta_t}, \quad \widehat{\phi}_t^B = 1 - \widehat{\phi}_t^A. \tag{11}$$

The number of shares of stock $\widehat{\theta}_t^A$ and units of riskless bonds $\widehat{\varphi}_t^A$ held by agent A are

$$\widehat{\theta}_t^A = \frac{\theta}{\theta + (1-\theta)\eta_t} - \frac{\theta(1-\theta)\delta_t\eta_t}{\left(\theta + (1-\theta)\eta_t\right)^2 \sigma\widehat{\sigma}_{S,t}} = \widehat{\phi}_t^A\left(1 - \frac{\widehat{\phi}_t^B\delta_t}{\sigma\widehat{\sigma}_{S,t}}\right), \tag{12}$$

$$\widehat{\varphi}_t^A = \frac{\widehat{\phi}_t^A\widehat{\phi}_t^B\delta_t}{\sigma\widehat{\sigma}_{S,t}}\widehat{S}_t, \tag{13}$$

and the corresponding values for agent B are $\theta_t^B = 1 - \theta_t^A$ and $\varphi_t^B = -\varphi_t^A$.

As Equation (12) shows, there are several forces affecting the portfolio positions. First, all else equal, agent A owns fewer shares of the stock when B has more optimistic beliefs (larger δ_t). This effect becomes weaker when the volatility of stock return $\widehat{\sigma}_{S,t}$ is high. Second, changes in the wealth distribution (as indicated by (11)) also affect the portfolio holdings, as the richer agent will tend to hold more shares of the stock.

We can gain more intuition on the stock price by rewriting Equation (9) as follows,

$$\widehat{S}_t = \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[D_T^{-1}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t^B\left[D_T^{-1}\right]} = \left(\frac{\widehat{\phi}_t^A}{\widehat{S}_t^A} + \frac{\widehat{\phi}_t^B}{\widehat{S}_t^B}\right)^{-1}, \tag{14}$$

which states that the stock price is a weighted harmonic average of the prices of the stock in two single-agent economies with agent A and B being the representative agent, $\widehat{S}_t^A$ and $\widehat{S}_t^B$, where

$$\widehat{S}_t^A = D_t e^{\left(\mu-\sigma^2\right)(T-t)}, \tag{15}$$

$$\widehat{S}_t^B = D_t e^{\left(\mu-\sigma^2\right)(T-t)-a(t,T)-b(t,T)\delta_t}, \tag{16}$$

and the weights $(\widehat{\phi}_t^A, \widehat{\phi}_t^B)$ are the two agents' shares of total wealth. For example, controlling for the wealth distribution, the equilibrium stock price is higher when agent B has more optimistic beliefs (larger δ_t).

One special case of the above result is when the amount of disagreement is constant over time ($\delta_t = \delta$ for all t). The results for this case are obtained by setting $\nu = 0$ and $\delta_0 = \bar{\delta} = \delta$ in Proposition 1. In particular, Equation (9) simplifies to

$$\widehat{S}_t = \frac{\theta + (1-\theta)\eta_t}{\theta + (1-\theta)\eta_t e^{-\delta(T-t)}} D_t e^{\left(\mu-\sigma^2\right)(T-t)}. \tag{17}$$

As another special case, the stock price in the case with no disagreement about the growth rate ($\delta_t = 0$ for all t) is

$$\widehat{S}_t = \widehat{S}_t^A = \frac{1}{\mathbb{E}_t[D_T^{-1}]} = D_t e^{\left(\mu-\sigma^2\right)(T-t)}, \tag{18}$$

which is a version of the Gordon growth formula, with σ^2 being the risk premium for the stock. The instantaneous volatility of stock returns becomes the same as the volatility of dividend growth, $\widehat{\sigma}_{S,t} = \sigma$. The shares of the aggregate stock held by the two agents will remain constant and be equal to the their endowments, $\widehat{\theta}_t^A = \theta, \widehat{\theta}_t^B = 1 - \theta$.

3.2 Circuit Breaker

In the presence of circuit breakers, there are two possible scenarios, (i) the circuit breaker is not triggered between 0 and T; (ii) the circuit breaker is triggered at time $\tau < T$. Thus, markets will remain dynamically complete over the interval $[0, \tau \wedge T]$, and we can still characterize the equilibrium allocation at $\tau \wedge T$ using the planner's problem. Our solution strategy is to solve for the optimal allocation at $\tau \wedge T$ for any exogenously given stopping time τ, and compute the corresponding stock price. The equilibrium is then the fixed point whereby the stopping time is consistent with the initial price S_0 (i.e., the stock price at the stopping time satisfies $S_\tau = (1 - \alpha)S_0$). Before doing so, we first characterize the agents' indirect utility function at the time of market closure when $\tau < T$.

Suppose agent i has wealth W_τ^i at time τ. Since the two agents behave competitively, they take the stock price S_τ as given and choose the shares of stock θ_τ^i and bonds φ_τ^i to maximize their expected utility over terminal wealth, subject to the budget constraint and the constraint on non-negative terminal wealth:

$$
\begin{aligned}
V^i(W_\tau^i, \tau) &= \max_{\theta_\tau^i, \varphi_\tau^i} \mathbb{E}_\tau^i \left[\ln(\theta_\tau^i D_T + \varphi_\tau^i) \right] \qquad (19) \\
s.t. \quad & \theta_\tau^i S_\tau + \varphi_\tau^i = W_\tau^i, \\
& W_T^i \geq 0,
\end{aligned}
$$

where $V^i(W_\tau^i, \tau)$ is the value function for agent i at the time when the circuit breaker is triggered.

The market clearing conditions are

$$\theta_\tau^A + \theta_\tau^B = 1, \tag{20a}$$

$$\varphi_\tau^A + \varphi_\tau^B = 0. \tag{20b}$$

For any $\tau < T$, the non-negative terminal wealth constraint implies that $\theta_\tau^i \geq 0, \varphi_\tau^i \geq 0$. That is, neither agent will take short or levered position in the stock. This result is due to the inability to rebalance one's portfolio after market closure, which is an extreme version of illiquidity. It then follows from the market clearing conditions above that in equilibrium neither agent will have any bond positions, which helps with the characterization of the equilibrium. The results are summarized in the following proposition.

Proposition 2. *Suppose the stock market closes at time $\tau < T$. In equilibrium, both agents will hold all of their wealth in the stock, $\theta_\tau^i = \frac{W_\tau^i}{S_\tau}$, and hold no bonds, $\varphi_\tau^i = 0$. The market clearing price is any price $S_\tau \leq S_\tau^*$,*

$$S_\tau^* = \min\{\widehat{S}_\tau^A, \widehat{S}_\tau^B\} = \begin{cases} D_\tau e^{\left(\mu-\sigma^2\right)(T-\tau)} & \text{if} \quad \delta_\tau > \underline{\delta}\left(\tau\right) \\ D_\tau e^{\left(\mu-\sigma^2\right)(T-\tau)-a(\tau,T)-b(\tau,T)\delta_\tau} & \text{if} \quad \delta_\tau \leq \underline{\delta}\left(\tau\right) \end{cases} \tag{21}$$

where $\widehat{S}_\tau^i$ denotes the stock price in a single-agent economy populated by agent i,

$$\underline{\delta}\left(t\right) = -\frac{a\left(t,T\right)}{b\left(t,T\right)}, \tag{22}$$

and $a(t,T)$, $b(t,T)$ are given in Proposition 1. Furthermore, $S_\tau = S_\tau^*$ is the unique equilibrium in the limit of a sequence of economies with bond supply Δ going to zero.

Proof. See Appendix A. □

Notice that the market clearing prices S_τ^* only depends on the belief of one of the two agents. Due to the non-negative wealth constraint, the market clearing price must be such that the relatively pessimistic agent (the one with lower valuation of the stock) is willing to invest all his wealth in the stock. Here, having the lower expectation of the growth rate at the current instant is not sufficient to make the agent marginal. One also needs to take into account the two agents' future beliefs and the risk premium associated with future fluctuations in the beliefs, which are summarized by $\underline{\delta}(t)$.

The price level S^*_τ is such that the marginal agent is indifferent between investing all his wealth or a bit more in the stock (no-leverage constraint is not binding). For any price $S_\tau < S^*_\tau$, the marginal investor would prefer to invest more than 100% of his wealth in the stock, but is prevented to do so by the no-leverage constraint. Thus, the market for the stock will still clear under these prices. These alternative equilibria can be ruled out if the net bond supply Δ is small but nonzero. This is because the relatively pessimistic agent will need to hold the bonds in equilibrium, which means his no-leverage constraint cannot be binding. Here on, we will focus our analysis on the case where $S_\tau = S^*_\tau$.

It follows from the definition of the circuit breaker and the continuity of stock prices that the stock price at the time of the trigger must satisfy $S_\tau = (1-\alpha)S_0$. This condition together with Equation (21) implies that we can characterize the stopping time τ using a stochastic threshold for dividend D_t.

Lemma 1. *Take the initial stock price S_0 as given. Define a stopping time*

$$\tau = \inf\{t \geq 0 : D_t = \underline{D}(t, \delta_t)\}, \tag{23}$$

where

$$\underline{D}(t, \delta_t) = \begin{cases} \alpha S_0 e^{-\left(\mu-\sigma^2\right)(T-t)} & \text{if} \quad \delta_t > \underline{\delta}(t) \\ \alpha S_0 e^{-\left(\mu-\sigma^2\right)(T-t)+a(t,T)+b(t,T)\delta_t} & \text{if} \quad \delta_t \leq \underline{\delta}(t) \end{cases} \tag{24}$$

Then, the circuit breaker is triggered at time τ when $\tau \leq T$.

Having characterized the equilibrium at time $\tau < T$, we plug the equilibrium portfolio holdings into (19) to derive the indirect utility of the two agents at τ,

$$V^i(W^i_\tau, \tau) = \mathbb{E}^i_\tau\left[\ln\left(\frac{W^i_\tau}{S_\tau}D_T\right)\right] = \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)]. \tag{25}$$

The indirect utility for agent i at $\tau \wedge T$ is then given by

$$V^i(W^i_{\tau\wedge T}, \tau \wedge T) = \begin{cases} \ln(W^i_T) & \text{if} \quad \tau \geq T \\ \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)] & \text{if} \quad \tau < T \end{cases} \tag{26}$$

These indirect utility functions make it convenient to solve for the equilibrium allocation in the economy with circuit breakers through the following planner problem,

$$\max_{W^A_{\tau\wedge T}, W^B_{\tau\wedge T}} \mathbb{E}_0 \left[\lambda V^A(W^A_{\tau\wedge T}, \tau \wedge T) + (1-\lambda)\eta_T V^B(W^B_{\tau\wedge T}, \tau \wedge T)\right], \tag{27}$$

subject to the resource constraint

$$W^A_{\tau\wedge T} + W^B_{\tau\wedge T} = S_{\tau\wedge T}, \tag{28}$$

where

$$S_{\tau\wedge T} = \begin{cases} D_T & \text{if} \quad \tau \geq T \\ (1-\alpha)S_0 & \text{if} \quad \tau < T \end{cases} \tag{29}$$

From the planner problem we get the wealth of agent A at time $\tau \wedge T$,

$$W^A_{\tau\wedge T} = \frac{\theta S_{\tau\wedge T}}{\theta + (1-\theta)\,\eta_{\tau\wedge T}}. \tag{30}$$

The SPD for agent A at time $\tau \wedge T$ is proportional to his marginal utility of wealth,

$$\pi^A_{\tau\wedge T} = \frac{\xi}{W^A_{\tau\wedge T}} = \frac{\xi\,(\theta + (1-\theta)\,\eta_{\tau\wedge T})}{\theta S_{\tau\wedge T}} \tag{31}$$

for some constant ξ. The price of the stock at time $t \leq \tau \wedge T$ is:

$$S_t = \mathbb{E}_t\left[\frac{\pi^A_{\tau\wedge T}}{\pi^A_t} S_{\tau\wedge T}\right] = \left(\phi^A_t \mathbb{E}_t[S^{-1}_{\tau\wedge T}] + \phi^B_t \mathbb{E}^B_t[S^{-1}_{\tau\wedge T}]\right)^{-1}, \tag{32}$$

where ϕ^i_t is the share of total wealth owned by agent i. The expectations in (32) are straightforward to evaluate numerically. Unlike in the case without circuit breakers, these expectations are not the inverse of the stock prices from the representative agent economies. This is because the thresholds and closing prices in the representative agent economies will be different from those in this economy.

From the stock price, one can then compute the conditional mean $\mu_{S,t}$ and volatility $\sigma_{S,t}$ of stock returns, which are given by

$$dS_t = \mu_{S,t} S_t dt + \sigma_{S,t} S_t dZ_t. \tag{33}$$

In Appendix A.3, we provide the closed-form solution for S_t in the special case with constant disagreements ($\delta_t = \delta$).

So far we have been taking the initial stock price S_0 as given when characterizing the threshold for the circuit breaker. Thus, the stock price S_t in (32) is a function of S_0 through its dependence on τ and $S_{\tau\wedge T}$, both of which depend on S_0. By evaluating S_t at time $t = 0$, we can finally solve for S_0 from the following fixed point problem,

$$S_0 = f(\tau(S_0), S_{\tau\wedge T}(S_0)), \tag{34}$$

where f is the function implied by Equation (32).

Proposition 3. *There is a unique solution to the fixed-point problem in* (34) *for any* $\alpha \in [0, 1]$.

Proof. See Appendix A. □

Finally, we examine the impact of circuit breakers on the wealth distribution. The wealth shares of the two agents at time $t \leq \tau \wedge T$ are the same as in the economy without circuit breakers,

$$\phi_t^A = \frac{W_t^A}{S_t} = \frac{\theta}{\theta + (1-\theta)\eta_t}, \quad \phi_t^B = 1 - \phi_t^A. \tag{35}$$

However, the wealth shares at the end of the trading day (time T) will be affected by the presence of circuit breakers. This is because if the circuit breaker is triggered at

$\tau < T$, the wealth distribution after τ will remain fixed due to the absence of trading. Since irrational traders on average lose money over time, market closure at $\tau < T$ will raise their average wealth share at time T. This "mean effect" implies that circuit breakers will help "protecting" the irrational investors in this model. How strong this effect is depends on the amount of disagreement and the distribution of τ. In addition, circuit breakers will also make the tail of the wealth share distribution thinner as they put a limit on the amount of wealth that the relatively optimistic investor can lose over time along those paths with low realizations of D_t.

4 Quantitative Analysis

We now analyze the quantitative implications of our model. First, in Section 4.1 we examine the special case of the model with constant disagreement, i.e., $\delta_t = \delta$ for all t, which is more transparent in demonstrating the main mechanism through which circuit breakers affect asset prices and trading. Then, in Section 4.2 we examine a version of the model with time-varying disagreements that mimic the behavior of investors facing leverage constraints.

4.1 Constant Disagreements

For calibration, we normalize $T = 1$ to denote one trading day. We set $\mu = 10\%/250 = 0.04\%$, which implies an annual dividend growth rate of 10%, and we assume daily volatility of dividend growth $\sigma = 3\%$. The circuit breaker threshold is set at 5% ($\alpha = 0.05$). For the initial wealth distribution, we assume agent A (with rational beliefs) owns 90% of total wealth ($\theta = 0.9$) at $t = 0$. Finally, for the amount of disagreement, we set $\delta = -2\%$. Thus, the irrational agent B is relatively pessimistic about dividend growth in this case.

In Figure 2, we plot the price-dividend ratio S_t/D_t (left column), the stock holding for agent A (middle column), and the bond holding for agent A (right column).

The stock and bond holdings for agent B can be inferred from those for agent A based on the market clearing conditions, namely $\theta_t^B = 1 - \theta_t^A$, $\varphi_t^B = -\varphi_t^A$. To demonstrate the time-of-day effect, we plot the solutions at three different points in time, $t = 0.25, 0.5, 0.75$. In each panel, the solid line denotes the solution for the case with circuit breakers, while the dotted line denotes the case without circuit breakers.

Let's start by examining the price-dividend ratio. As shown in (14), the price of the stock in the case without circuit breakers is the weighted (harmonic) average of the prices of the stock from the two representative-agent economies populated by agent A and B, respectively, with the weights given by the two agents' shares of total wealth. Under our calibration, $\mu - \sigma^2$ is very close to zero, implying that the price-dividend ratio is close to one for any $t \in [0, T]$ in the economy with agent A only (denoted by the upper horizontal dash lines in the first column), while it is approximately $e^{\delta(T-t)} \leq 1$ in the economy with agent B only (denoted by the lower horizontal dash lines in the first column). Thus, the gap between the price-dividend ratios from the two representative-agent economies will be at most about 2% (since $\delta = -2\%$) under our calibration, which is fairly modest.

As the left column of Figure 2 shows, the price-dividend ratio in the economy without circuit breakers (red dotted line) indeed lies between the price-dividend ratios $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ for the two representative-agent economies (and thus always below 1). Since agent A is relatively more optimistic, he chooses to hold levered position in the stock (see the plots for θ_t^A in the middle column of Figure 2), and his share of total wealth will become higher following positive shocks to the dividend. As a result, as dividend value D_t rises (falls), the share of total wealth owned by agent A increases (decreases), which make the equilibrium price-dividend ratio approach the value $\widehat{S}_t^A/D_t$ ($\widehat{S}_t^B/D_t$). Finally, the price-dividend ratio becomes higher (converges to one) as t approaches T, as shorter horizon reduces the impact of agent B is pessimistic beliefs on stock price.

In the case with circuit breakers, the price-dividend ratio (blue solid line) still lies between the price-dividend ratios from the two representative agent economies, but it is always below the price-dividend ratio without circuit breakers for a given level of dividend. The gap between the two price-dividend ratios is negligible when D_t is sufficiently high, but it widens as D_t approaches the circuit breaker threshold $\underline{D}(t)$.

The reason that the stock price declines more rapidly with dividend in the presence of circuit breakers is as follows. As explained in Section 3.2, upon triggering the circuit breaker, neither agent (agent A in particular) will be willing to take a levered position in the stock due to the complete illiquidity of the stock. As a result, the relatively pessimistic agent (agent B in this case) becomes the marginal investor. That is, the market clearing stock price has to be such that agent B is willing to hold all of his wealth in the stock, regardless of his share of total wealth. As a result, we see the price-dividend ratio with circuit breakers converges to $\widehat{S}_t^B/D_t$ when $D_t = \underline{D}(t)$, while the price-dividend ratio without circuit breakers is still a weighted average of $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ at this point. The lower equilibrium stock valuation at the circuit breaker threshold also drives the value of the stock lower before reaching the threshold, although the effect dissipates as we move further away from the threshold. Furthermore, because the gap between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ shrinks as t approaches T, the impact of circuit breakers on the price-dividend ratio also becomes smaller as t increases. For example, at $t = 0.25$, the price-dividend ratio with circuit breakers can be as much as 1.2% lower than the level without circuit breakers. At $t = 0.75$, the gap is at most 0.3%.

We can also analyze the impact of the circuit breakers on the equilibrium stock price by connecting it to how circuit breakers influence the equilibrium portfolio holdings of the two agents. Let us again start with the case without circuit breakers (red dotted lines in middle and right columns of Figure 2). The stock holding of agent A (θ_t^A) continues to rise as D_t falls to $\underline{D}(t)$ and beyond. This is the result of two effects: (i) with lower D_t, the stock price is lower, implying higher expected return under agent A's beliefs; (ii) lower D_t also makes agent B (who is shorting the stock) wealthier and thus more capable of lending to agent A to take on a levered position (ϕ_t^A becomes more negative).

With circuit breakers, while the stock position θ_t^A takes on similar values for large values of D_t, it becomes substantially lower than the value without circuit breakers when D_t is sufficiently close to the circuit breaker threshold, and can eventually become decreasing as D_t drops. Accordingly, the amount of borrowing by agent A $(-\phi_t^A)$ also eventually decreases as D_t drops. This is because agent A becomes increasingly concerned with the rising return volatility at lower D_t, which dominates the effect of the increase in the expected stock return (we analyze the expected return and volatility in more detail in Figure 3). Finally, θ_t^A takes a discrete drop to 1 when $D_t = \underline{D}(t)$. The cutting back of stock position (deleveraging) by agent A before the circuit breaker is triggered can be interpreted as a form of "self-predatory" trading, and the stock price in equilibrium has to fall enough such that agent A has no incentive to sell more of his stock holdings.

Next, Figure 3 shows the conditional return volatility and the conditional expected returns under the two agents' beliefs. Compared to the relatively modest effects on stock price levels, the impact of circuit breakers on the conditional volatility of stock returns can be much more sizable. Without circuit breakers, the conditional volatility of returns (red dotted lines in the left column) peaks at about 3.2%, only slightly higher than the fundamental volatility of $\sigma = 3\%$. The excess volatility comes from the time variation in the wealth distribution between the two agents, which peaks when the wealth shares of the two agents are about equal and is small in magnitude.

With circuit breakers, the conditional return volatilities (blue solid lines in the left column) are close in value to the conditional return volatilities without circuit breakers when D_t is high, but it becomes substantially higher as D_t approaches $\underline{D}(t)$. Furthermore, the volatility amplification effect of the circuit breaker becomes stronger when the circuit breaker is triggered earlier during the day. When $t = 0.25$, the conditional volatility reaches 6% at the circuit breaker threshold, almost twice as high as the return volatility without circuit breakers. When $t = 0.75$, the conditional volatility peaks at 4.5% at the circuit breaker threshold, compared to the return volatility of 3% without circuit breakers.

The middle column of Figure 3 show the conditional expected returns for the stock under agent A's beliefs (the objective probability measure). Again, we start with the case without circuit breakers. The expected returns (red dotted lines) are positive for the rational agent (A) and negative for the pessimist (agent B). Under agent A's beliefs, the expected return is higher when D_t is lower. This is because his share of total wealth is smaller and his leverage ratio is higher when the level of dividend is lower. As a result, his marginal value of wealth becomes more volatile, which leads to higher conditional Sharpe ratio. The higher Sharpe ratio, combined with relatively stable conditional return volatility, then implies higher expected returns for the stock.

In the presence of circuit breakers, the conditional Sharpe ratio for agent A rises even faster with lower D_t than without circuit breakers. Since the conditional return volatility also rises faster in this case, we get the conditional expected return (blue solid lines) under agent A's beliefs to rise more rapidly with lower D_t. However, as D_t gets sufficiently close to $\underline{D}(t)$, the threat of market closure leads agent A to reduce his stock holdings since the increase in conditional volatility is large enough to dominate the effect of higher conditional expected return.

We now to turn the impact of circuit breakers on the wealth distribution. As shown in Equation (35), at any time before market closure ($t < \tau$), the wealth shares of the two agents (ϕ_t^A, ϕ_t^B) are only functions of η_t and will thus be identical to the wealth shares in the economy without circuit breakers (given the same initial wealth distribution and the same history of dividend shocks). However, after market closure, the wealth shares in the economy with circuit breakers will remain fixed, whereas the wealth shares in the economy without circuit breakers will continue to evolve due to dynamic trading. As a result, the wealth distribution at the end of the trading day T can become different with circuit breakers.

In particular, the mean of the terminal wealth share for the irrational (rational) agent should become higher (lower) with circuit breakers, as market closure helps protect the irrational agents by preventing them from betting on the wrong beliefs. Additional differences in the terminal wealth distribution depend on the distribution of τ, specifically how frequently and when the circuit breaker will be triggered.

In Figure 4, we plot the distribution of terminal wealth share for the two agents without (red dotted line) and with (blue solid line) circuit breakers. Recall that agent A starts the trading day with 90% total wealth. In the economy without circuit breakers, the distribution of agent A's terminal wealth share, $\widehat{\phi}_T^A$, is unimodal, with the mean (0.9043) and the median (0.94) above 0.9 due to the fact that agent A has correct beliefs and gains wealth on average from agent B.

With circuit breakers, the distribution of agent A's terminal wealth share, ϕ_T^A, does have a lower mean (0.9036). In addition, the distribution becomes bimodal, with the new mode resulting from the circuit breaker trigger. Notice that the new mode is more than 10% below the initial wealth share, which is a significant amount of loss in wealth for agent A in just one trading day. In addition, the left tail of the distribution for ϕ_T^A becomes thinner. Thus, the presence of circuit breakers eliminates the possibility of extreme losses for the rational agent, but at the cost of substantially increasing the likelihood of significant losses. Correspondingly, circuit breakers will increase the likelihood of significant gains in wealth for the irrational agent B. These states with significant wealth gains for agent B are also the states with large price distortions. Thus, while circuit breakers do protect the irrational agents from losing too much wealth, which might be a justifiable objective for a paternalistic social planner (regulator), our model shows an additional side effect of circuit breakers on price distortion.

Through its impact on the conditional price-dividend ratio, conditional volatility, and the wealth distribution post market closure, circuit breakers also affect the distribution of daily average price-dividend ratios, daily price ranges (defined as daily high minus low prices), and daily return volatilities (the square root of the quadratic variation of $\log(S_t)$ over $[0, T]$). We examine these effects in Figure 5. The top panel shows that the distribution of daily average price-dividend ratio is shifted to the left in the presence of circuit breakers, which shows that circuit breakers indeed lead to more downside price distortion in this model.

Next, the results for the daily price range distribution show that circuit breakers can reduce the probabilities of having very large daily price ranges (those over 6.5%), but they would raise the probabilities of daily price ranges between 4.5 and 6.5%. Moreover, circuit breakers generate significant fatter tails for the distributions of daily realized volatilities (in addition to the larger conditional volatilities shown earlier).[5] The results of these two volatility measures both show that the effect of circuit breakers on return volatility is far more intricate than what the naive intuition would suggest.

"The Magnet Effect" "The magnet effect" is a phenomenon that is often associated with circuit breakers and price limits. Informally, it refers to the acceleration of price movement towards the circuit breaker threshold (price limit) as the price approaches the threshold (limit). We try to formalize this notion in our model by considering the conditional probability that the stock price, currently at S_t, will reach the circuit breaker threshold $(1-\alpha)S_0$ within a given period of time h. In the case without circuit breakers, we can again compute the probability of the stock price reaching the same threshold over the period h. As we will show, our version of "magnet effect" refers to the fact that the very presence of circuit breakers increases the probability of the stock price reaching the threshold in a short period of time, with this effect becoming stronger when the stock price is close to the threshold.

In Figure 6, we consider three different horizons, $h = 10, 30, 90$ minutes. When S_t is sufficiently far from $(1-\alpha)S_0$, the gap between the conditional probabilities with and without circuit breakers indeed widens as the stock price moves closer to the threshold, which is consistent with the "magnet effect" defined above. This effect is caused by the significant increase in conditional return volatility in the presence of circuit breakers. However, the gap between the two conditional probabilities eventually starts to narrow, because both probabilities will converge to 1 as S_t reaches $(1-\alpha)S_0$. Looking at different horizons h, we see that the largest gap in the two conditional probabilities occurs closer to the threshold when h is small. Moreover, the increase in

[5]Realized volatility is measured as the square root of the quadratic variation in log price.

probability of reaching the threshold is larger earlier during the trading day.

The volatility amplification effect and the initial wealth share θ In our benchmark calibration, we assume the rational agent initially owns $\theta = 90\%$ of total wealth. It appears quite intuitive that allocating more wealth initially (smaller θ) to the irrational agent should increase their price impact through the trading day. While this intuition is correct for the price level, the effects on return volatility is just the opposite, especially near the circuit breaker threshold.

In the left panel of Figure 7, we examine the the volatility amplification effect of circuit breakers by plotting the ratio of conditional return volatilities for the cases with and without circuit breakers at the threshold $\underline{D}(t)$ as a function of θ. As the graph shows, the amplification effect is in fact stronger the less wealth the irrational agent owns initially.

This seemingly counterintuitive result is due to the extreme illiquidity caused by market closure. Recall that the stock price with circuit breakers will be close to the price without circuit breakers when D_t is sufficiently far away from $\underline{D}(t)$, and the latter is a wealth-share weighted average of the price-dividend ratios under the two agents' beliefs. Upon triggering the circuit breaker, the inability to rebalance the portfolio prevents the rational agent A from holding a levered position in the stock, and the stock price has to fully agree with the pessimistic agent B's beliefs in order to clear the market. Thus, the less wealth the irrational agent has initially, the more abruptly the stock price's change will be from mostly reflecting the beliefs of agent A (when D_t is large) to entirely reflecting the beliefs of agent B (at $\underline{D}(t)$). This elevated sensitivity of stock price to changes in D_t then translates into a stronger volatility amplification effect.

The impact of circuit breaker limit α So far we have considered a single level of circuit breaker limit of $\alpha = 5\%$. In Figure 8, we examine how different levels of circuit breaker limits affect the distributions of price-dividend ratios and volatility. We see that tighter circuit breaker thresholds tend to reduce price-dividend ratios, which corresponds to larger price distortions. Furthermore, while tighter circuit breaker thresholds do tend to reduce daily price ranges, they can significantly increase realized volatilities.

Finally, in Panel D, we examine the probability of triggering the circuit breaker within the trading day as the limit changes. The probability of the stock price falling to a given threshold is a nonlinear function of the limit. Importantly, this probability rises more rapidly as the threshold becomes tighter when there is a circuit breaker than when there is not. This result suggests that using historical data from a period when circuit breakers have not been implemented can lead one to underestimate the likelihood of triggering a circuit breaker, especially when the threshold is relatively tight.

4.2 Time-varying Disagreements

So far we have been studying the setting with constant disagreement between the two agents. To illustrate the impact of time-varying disagreements, we consider the case where the difference in beliefs δ_t follows a random walk, i.e., $\kappa = 0$ and $\nu = 0.5\sigma$, and we set $\delta_0 = 0$ such that initially there is no bias in agent B's belief.

Agent B's belief under this calibration resembles the "representativeness" bias in the behavioral finance literature. He adjusts his belief excessively by extrapolating short-term trends in fundamentals, becoming overly optimistic following large positive dividend shocks and panicking following large negative dividend shocks. Such beliefs could also represent the behavior of constrained investors, who become effectively more (less) pessimistic or risk averse as the constraint tightens (loosens).

Figure 9 shows the price-dividend ratio and portfolio holdings, while Figure 10 shows the conditional return volatility and expected returns. We again start with the results for the case without circuit breakers. First, the price-dividend ratio has a wider range than in the constant disagreement case. Agent B becomes more optimistic (pessimistic) as D_t rises (falls) from the initial level, and δ_t can take on a wide range of values as a result. Naturally, the effect of time-varying disagreement becomes more limited as t approaches T, and the price-dividend ratio eventually converges to 1.

Second, the conditional return volatility now reflects the joint effects of the wealth distribution and the changes in the amount of disagreement. It not only tends to become higher when the distribution of wealth between the two agents is more even, but also when the amount of disagreement is high (large $|\delta_t|$). The result is a U-shaped conditional volatility as function of D_t, which is more pronounced when t is small.

Next, in the presence of circuit breakers, when D_t is above the circuit breaker threshold, the price-dividend ratio is again lower than the value without circuit breakers. This is consistent with our finding in the constant disagreement case. However, the circuit breaker does rule out some extreme values for the price-dividend ratio during the trading session. This result could have significant consequences when there are intra-day mark-to-market requirements for some of the market participants, as a narrower range of price-dividend ratio can help lower the chances of forced liquidation when the stock price falls by a large amount. Thus, one of the benefits of having circuit breakers could be to limit the range of price-dividend ratio in this case.

Similar to the constant disagreement case, circuit breakers in the case of time-varying disagreement amplify the conditional volatility by a significant amount when D_t is close to the threshold $\underline{D}_t$, especially when t is small. A difference is that the conditional volatility is lower with circuit breakers than without when D_t is sufficiently large. As D_t becomes large, so does δ_t, which makes return more volatile. However, the optimistic agent B is made less aggressive by the threat of market closure and loss of liquidity, which is why volatility is dampened in the presence of circuit breakers.

5 Upper and Lower Boundary

In this section we extend our analysis to the case of the two-sided circuit breakers; i.e. the circuit breaker is triggered when the stock price reaches either $(1 - \alpha^D) S_0$

or $(1+\alpha^U)\,S_0$ (whatever happens first); $\alpha^U, \alpha^D > 0$. We assume that agents have constant disagreement in beliefs and consider the same numerical parameters as in Section 4.1 with $\alpha^U = \alpha^D = 5\%$.

Figure 11 is the counterpart of Figure 2 for the two-sided circuit breaker case. As before in the left column we present the plots of the price-dividend ratio S_t/D_t as a function of D_t. In the two-sided circuit breaker case the S_t/D_t ratio still lies between the price-dividend ratios $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ for the two representative-agent economies. Now, however, it has an inverse 'U' shape. For small and intermediary values of D_t price-dividend ratio is an increasing function of D_t. As in one-sided circuit breaker case S_t/D_t equals $\widehat{S}_t^B/D_t$ when $D_t = \underline{D}(t)$ and approaches the price-dividend ratio in the economy without circuit breakers for intermediate values of D_t. For higher values of D_t though, S_t/D_t decreases with D_t and again equals $\widehat{S}_t^B/D_t$ when the dividend reaches the boundary value that triggers the circuit breaker. Interpretation of this behavior of the price-dividend ratio for high values of D_t is similar to the one-sided circuit breaker case. When the circuit breaker is triggered none of the agents is willing to hold either leveraged or short position in stock. In this situation the most pessimistic agent is the marginal investor regardless of whether the stock price has reached the upper or lower boundary.

This behavior of the price-dividend ratio manifests itself in the dependence of conditional volatility on dividend presented in the left column of Figure 12 (the counterpart of Figure 3 in the one-sided circuit breakers case). Conditional volatility is the highest for low values of D_t when S_t/D_t is increasing in D_t. Conditional volatility gradually decreases with D_t as the slope of S_t/D_t decreases. Finally conditional volatility approaches its lowest value when D_t equals its highest (boundary) value.

This analysis reveals similarities and differences between circuit breakers imposed on the stock price from below and circuit breakers imposed on the stock price from above. In both cases in the presence of circuit breakers price-dividend ratio declines relative to the complete markets economy. The impact on volatility is asymmetric though: lower price boundary tends to amplify volatility, while upper boundary dampens it.

6 Conclusion

In this paper, we build a dynamic model to examine the mechanism through which market-wide circuit breakers affect trading and price dynamics in the stock market. As we show, circuit breakers tends to lower the price-dividend ratio, reduce daily price ranges, but increase conditional and realized volatilities. They also raise the probability of the stock price reaching the circuit breaker limit as the price approaches the threshold (a "magnet" effect). The effects of circuit breakers can be further amplified when some agents' willingness to hold the stock is sensitive to recent shocks to fundamentals, which can be due to behavioral biases, institutional constraints, etc. Finally, using historical data from a period when circuit breakers have not been implemented can lead one to underestimate the likelihood of triggering a circuit breaker, especially when the threshold is relatively tight.

Appendix

A Proofs

A.1 Proof of Proposition 1

When there are no circuit breakers, the stock price is

$$\widehat{S}_t = \mathbb{E}_t\left[\frac{\widehat{\pi}_T^A D_T}{\mathbb{E}_t\left[\widehat{\pi}_T^A\right]}\right] = \frac{\mathbb{E}_t\left[\theta + (1-\theta)\,\eta_T\right]}{\mathbb{E}_t\left[D_T^{-1}\left(\theta + (1-\theta)\,\eta_T\right)\right]} = \frac{\theta + (1-\theta)\,\eta_t}{\theta\mathbb{E}_t\left[D_T^{-1}\right] + (1-\theta)\,\mathbb{E}_t\left[D_T^{-1}\eta_T\right]}, \quad (36)$$

where

$$\mathbb{E}_t\left[D_T^{-1}\right] = D_t^{-1} e^{-\left(\mu-\sigma^2\right)(T-t)}, \quad (37)$$

and

$$\mathbb{E}_t\left[D_T^{-1}\eta_T\right] = \eta_t \mathbb{E}_t\left[D_T^{-1}\frac{\eta_T}{\eta_t}\right] = \eta_t \mathbb{E}_t^B\left[D_T^{-1}\right]. \quad (38)$$

Define the log dividend $x_t = \log D_t$. Under measure $\mathbb{P}^B$, the processes for x_t and δ_t are

$$dx_t = \left(\mu - \frac{\sigma^2}{2} + \delta_t\right) dt + \sigma dZ_t^B, \quad (39a)$$

$$d\delta_t = \left(\kappa\bar{\delta} + \left(\frac{\nu}{\sigma} - \kappa\right)\delta_t\right) dt + \nu dZ_t^B. \quad (39b)$$

Define $X_t = [x_t \ \delta_t]'$, then X_t follows an affine process,

$$dX_t = \left(K_0 + K_1 X_t\right) dt + \sigma_X dZ_t^B, \quad (40)$$

with

$$K_0 = \begin{bmatrix} \mu - \frac{\sigma^2}{2} \\ \kappa\bar{\delta} \end{bmatrix}, \ K_1 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{\nu}{\sigma} - \kappa \end{bmatrix}, \ \sigma_X = \begin{bmatrix} \sigma \\ \nu \end{bmatrix}. \quad (41)$$

We are interested in computing

$$g\left(t, X_t\right) = \mathbb{E}_t^B\left[e^{\rho_1' X_T}\right], \quad \text{with } \rho_1 = [-1 \ \ 0]'. \quad (42)$$

By applying standard results for the conditional moment-generating functions of affine processes (see, .e.g., Singleton 2006), we get

$$g\left(t,X_t\right)=\exp\left(A\left(t,T\right)+B\left(t,T\right)'X_t\right), \tag{43}$$

where

$$\begin{aligned} 0 &= \dot{B}+K_1'B, \quad B\left(T,T\right)=\rho_1 && \text{(44a)} \\ 0 &= \dot{A}+B'K_0+\frac{1}{2}tr\left(BB'\sigma_X\sigma_X'\right), \quad A\left(T,T\right)=0 && \text{(44b)} \end{aligned}$$

Solving for the ODEs gives

$$B\left(t,T\right)=\left[-1 \quad \frac{1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}}{\frac{\nu}{\sigma}-\kappa}\right]', \tag{45}$$

and

$$\begin{aligned} A\left(t,T\right)=&\left(\mu-\sigma^2-\frac{\kappa\overline{\delta}-\sigma\nu}{\frac{\nu}{\sigma}-\kappa}-\frac{\nu^2}{2\left(\frac{\nu}{\sigma}-\kappa\right)^2}\right)\left(t-T\right)-\frac{\nu^2}{4\left(\frac{\nu}{\sigma}-\kappa\right)^3}\left(1-e^{2\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right) \\ &+\left(\frac{\kappa\overline{\delta}-\sigma\nu}{\left(\frac{\nu}{\sigma}-\kappa\right)^2}+\frac{\nu^2}{\left(\frac{\nu}{\sigma}-\kappa\right)^3}\right)\left(1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right) \end{aligned} \tag{46}$$

After plugging the above results back into (36) and reorganizing the terms, we get

$$S_t=\frac{\theta+\left(1-\theta\right)\eta_t}{\theta+\left(1-\theta\right)\eta_t H\left(t,\delta_t\right)}D_t e^{\left(\mu-\sigma^2\right)(T-t)}, \tag{47}$$

where

$$\begin{aligned} H\left(t,\delta_t\right) &= e^{a(t,T)+b(t,T)\delta_t}, && \text{(48a)} \\ a\left(t,T\right) &= \left(\frac{\kappa\overline{\delta}-\sigma\nu}{\frac{\nu}{\sigma}-\kappa}+\frac{\nu^2}{2\left(\frac{\nu}{\sigma}-\kappa\right)^2}\right)\left(T-t\right)-\frac{\nu^2}{4\left(\frac{\nu}{\sigma}-\kappa\right)^3}\left(1-e^{2\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right) \\ &\quad+\left(\frac{\kappa\overline{\delta}-\sigma\nu}{\left(\frac{\nu}{\sigma}-\kappa\right)^2}+\frac{\nu^2}{\left(\frac{\nu}{\sigma}-\kappa\right)^3}\right)\left(1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right), && \text{(48b)} \\ b\left(t,T\right) &= \frac{1-e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}}{\frac{\nu}{\sigma}-\kappa}. && \text{(48c)} \end{aligned}$$

Finally, to compute the conditional volatility of stock returns, we have

$$\begin{aligned} d\widehat{S}_t &= \widehat{\mu}_{S,t}\widehat{S}_t dt + \widehat{\sigma}_{S,t}\widehat{S} dZ_t \\ &= o(dt) + \widehat{S}_t \frac{dD_t}{D_t} \\ &\quad + \eta_t D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{\theta(1-\theta)(1-H(t,\delta_t))}{(\theta+(1-\theta)\eta_t H(t,\delta_t))^2} \frac{d\eta_t}{\eta_t} \\ &\quad - D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{(\theta+(1-\theta)\eta_t)(1-\theta)\eta_t H(t,\delta_t) b(t,T)}{(\theta+(1-\theta)\eta_t H(t,\delta_t))^2} d\delta_t. \end{aligned}$$

After collecting the diffusion terms, we get

$$\begin{aligned} \widehat{\sigma}_{S,t} = \sigma + \frac{D_t e^{\left(\mu-\sigma^2\right)(T-t)}}{\widehat{S}_t} \Bigg(& \frac{\theta(1-\theta)(1-H(t,\delta_t))}{(\theta+(1-\theta)\eta_t H(t,\delta_t))^2} \frac{\delta_t \eta_t}{\sigma} \\ & - \frac{(\theta+(1-\theta)\eta_t)(1-\theta)\eta_t b(t,T) H(t,\delta_t)}{(\theta+(1-\theta)\eta_t H(t,\delta_t))^2} \nu \Bigg). \end{aligned} \quad (49)$$

A.2 Proof of Proposition 2

Suppose market closes at time τ. Since bonds are zero net supply,

$$W_{1,\tau} + W_{2,\tau} = S_\tau$$

Then the agents' problems at time τ are

$$\begin{aligned} V^1(W_{1,\tau}) &= \max_{\theta_1,b_1} E_\tau^1\left[\ln(\theta_1 D_T + b_1)\right] \\ s.t. \quad \theta_1 S_\tau + b_1 &\leq W_{1,\tau} \\ \theta_1 \geq 0, &\qquad b_1 \geq 0 \end{aligned}$$

and

$$\begin{aligned} V^2(W_{2,\tau}) &= \max_{\theta_2,b_2} E_\tau^2\left[\ln(\theta_2 D_T + b_2)\right] \\ s.t. \quad \theta_2 S_\tau + b_2 &\leq W_{2,\tau} \\ \theta_2 \geq 0, &\qquad b_2 \geq 0 \end{aligned}$$

The Lagrangian:

$$L = E_\tau^1 \left[\ln \left(\theta_1 D_T + b_1\right)\right] + \zeta^1 \left(W_{1,\tau} - \theta_1 S_\tau - b_1\right) + \xi_a^1 \theta_1 + \xi_b^1 b_1$$

FOC:

$$\begin{aligned} E_\tau^i \left[\frac{D_T}{\theta_i D_T + b_i}\right] - \zeta^i S_\tau + \xi_a^i &= 0 \\ E_\tau^i \left[\frac{1}{\theta_i D_T + b_i}\right] - \zeta^i + \xi_b^i &= 0 \end{aligned}$$

Suppose agent 1 is less optimistic than agent 2. Then we can quickly examine following three cases:

1. It could be an equilibrium when the price is sufficiently low such that both agents want to take levered positions (putting more than 100% of their wealth in the stock) but are constrained from borrowing. In this case, both agents submit demands proportional to their wealth,

$$\begin{aligned} \theta_i^* &= \frac{W_{i,\tau}}{S_\tau} \\ \xi_a^1 &= \xi_a^2 = 0 \\ \xi_b^1 &> 0, \xi_b^2 > 0 \end{aligned}$$

and the market for the stock clears. In this case,

$$S_\tau = \frac{E_\tau^1 \left[\frac{D_T}{\theta_1 D_T + b_1}\right]}{E_\tau^1 \left[\frac{1}{\theta_1 D_T + b_1}\right] + \xi_b^1} = \frac{1}{E_\tau^1 \left[\frac{1}{D_T}\right] + \xi_b^1 \theta_1} < \frac{1}{E_\tau^1 \left[\frac{1}{D_T}\right]} = S_\tau^*.$$

2. It could be an equilibrium when agent 1 (pessimist) finds it optimal to hold all his wealth in the stock, while agent 2 (optimist) is constrained from borrowing:

$$\begin{aligned} \theta_i^* &= \frac{W_{i,\tau}}{S_\tau} \\ \xi_a^1 &= \xi_a^2 = 0 \\ \xi_b^1 &= 0, \xi_b^2 > 0 \end{aligned}$$

Then from agent 1,

$$\begin{aligned} E_\tau^1\left[\frac{D_T}{\theta_1 D_T + b_1}\right] - \zeta^1 S_\tau &= 0 \\ E_\tau^1\left[\frac{1}{\theta_1 D_T + b_1}\right] - \zeta^1 &= 0 \end{aligned}$$

This implies

$$S_\tau = \frac{E_\tau^1\left[\frac{D_T}{\theta_1^* D_T + b_1^*}\right]}{E_\tau^1\left[\frac{1}{\theta_1^* D_T + b_1^*}\right]} = \frac{E_\tau^1\left[\frac{1}{\theta_1^*}\right]}{E_\tau^1\left[\frac{1}{\theta_1^* D_T}\right]} = \frac{1}{E_\tau^1\left[\frac{1}{D_T}\right]} = D_\tau e^{\left(\mu-\sigma^2\right)(T-\tau)} = S_\tau^*$$

The latter follows from the fact that market clearing for bonds implies that $b_1^* = b_2^* = 0$,and $\theta_1^* > 0$ as long as $W_{1,\tau} > 0$. Let's check whether this is this consistent with agent 2.

$$\begin{aligned} E_\tau^2\left[\frac{D_T}{\theta_2 D_T + b_1}\right] - \zeta^2 S_\tau &= 0 \\ E_\tau^2\left[\frac{1}{\theta_2 D_T + b_2}\right] - \zeta^2 + \xi_b^2 &= 0 \end{aligned}$$

$$S_\tau = \frac{E_\tau^2\left[\frac{D_T}{\theta_2^* D_T + b_2^*}\right]}{E_\tau^2\left[\frac{1}{\theta_2^* D_T + b_2^*}\right] + \xi_b^2} = \frac{E_\tau^2\left[\frac{1}{\theta_2^*}\right]}{E_\tau^2\left[\frac{1}{\theta_2^* D_T}\right] + \xi_b^2} = \frac{1}{E_\tau^2\left[\frac{1}{D_T}\right] + \xi_b^2 \theta_2^*}.$$

Since agent 2 is more optimistic,

$$E_\tau^1\left[\frac{1}{D_T}\right] > E_\tau^2\left[\frac{1}{D_T}\right],$$

which implies

$$\xi_b^2 = \frac{E_\tau^1\left[\frac{1}{D_T}\right] - E_\tau^2\left[\frac{1}{D_T}\right]}{\theta_2^*} > 0.$$

3. For any $S_\tau > S_\tau^*$, agent 1 will prefer to hold less than 100% of the wealth in the stock. This would require agent 2 to take levered position, which cannot be an equilibrium.

We will restrict our attention to equilibriums of type 2.

A.3 Special Case: Constant Disagreement

The stock price can be computed in closed form in the case of constant disagreement, $\delta_t = \delta$. Without loss of generality, we focus on the case where agent B is relatively more optimistic, $\delta \geq 0$. The results are summarized below.

Proposition 4. *Take S_0 as given. With $\delta \geq 0$, the stock price time $t \leq \tau \wedge T$ is*

$$S_t = \left(\phi_t^A \mathbb{E}_t[S_{\tau\wedge T}^{-1}] + \phi_t^B \mathbb{E}_t^B[S_{\tau\wedge T}^{-1}]\right)^{-1}, \tag{50}$$

where

$$\begin{aligned}
\mathbb{E}_t[S_{\tau\wedge T}^{-1}] &= \frac{1}{\alpha S_0}\left(N\left(\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right) + e^{\sigma d_t} N\left(\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right)\right) \\
&\quad + D_t^{-1} e^{-\left(\mu-\sigma^2\right)(T-t)}\left(N\left(-\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right) - e^{-\sigma d_t} N\left(\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right)\right), \qquad (51) \\
\mathbb{E}_t^B[S_{\tau\wedge T}^{-1}] &= \frac{1}{\alpha S_0}\left(N\left(\frac{d_t - \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right) + e^{\left(\sigma + \frac{2\delta}{\sigma}\right)d_t} N\left(\frac{d_t + \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right)\right) \\
&\quad + D_t^{-1} e^{-\left(\mu-\sigma^2+\delta\right)(T-t)}\left(N\left(-\frac{d_t - \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right)\right. \\
&\quad \left. - e^{\left(\frac{2\delta}{\sigma} - \sigma\right)d_t} N\left(\frac{d_t + \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right)\right), \qquad (52)
\end{aligned}$$

and

$$d_t = \frac{1}{\sigma}\left(\log\left(\frac{\alpha S_0}{D_t}\right) - \left(\mu - \sigma^2\right)(T-t)\right). \tag{53}$$

Proof. As show in Section 3.2, the stock price at time $t \leq \tau \wedge T$ is

$$\begin{aligned}
S_t &= \frac{\mathbb{E}_t\left[\pi_{\tau\wedge T}^A S_{\tau\wedge T}\right]}{\pi_t^A} = \frac{\theta + (1-\theta)\eta_t}{\mathbb{E}_t\left[\frac{\theta + (1-\theta)\eta_{\tau\wedge T}}{S_{\tau\wedge T}}\right]} \\
&= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S_{\tau\wedge T}^{-1}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[\frac{\eta_{\tau\wedge T}}{\eta_t} S_{\tau\wedge T}^{-1}\right]} \\
&= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S_{\tau\wedge T}^{-1}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t^B\left[S_{\tau\wedge T}^{-1}\right]}. \qquad (54)
\end{aligned}$$

The second equality follows from Doob's Optional Sampling Theorem, while the last equality follows from Girsanov's Theorem.

Now consider the case when $\delta_t = \delta$. Taking S_0 as given and imposing the condition for stock price at the circuit breaker trigger, we have

$$\mathbb{E}_t\left[S_{\tau\wedge T}^{-1}\right] = \frac{1}{\alpha S_0} P_t(\tau \leq T) + \mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right], \tag{55}$$

$$\mathbb{E}_t^B\left[S_{\tau\wedge T}^{-1}\right] = \frac{1}{\alpha S_0 \eta_t}\mathbb{E}_t\left[\eta_\tau \mathbb{1}_{\{\tau\leq T\}}\right] + \frac{1}{\eta_t}\mathbb{E}_t\left[\eta_T D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right]. \tag{56}$$

The following standard results about hitting times of Brownian motions are helpful for deriving the expressions for the expectations in (55)-(56) (see e.g., Jeanblanc, Yor, and Chesney, 2009, chap 3). Let Z^μ denote a drifted Brownian motion, $Z_t^\mu = \mu t + Z_t$, with $Z_0^\mu = 0$. Let $\mathcal{T}_y^\mu = \inf\{t \geq 0 : Z_t^\mu = y\}$ for $y < 0$. Then

$$\Pr\left(T_y^\mu \leq t\right) = N\left(\frac{y - \mu t}{\sqrt{t}}\right) + e^{2\mu y} N\left(\frac{y + \mu t}{\sqrt{t}}\right), \tag{57}$$

$$E\left[e^{-\lambda T_y^\mu}\mathbb{1}_{\{T_y^\mu \leq t\}}\right] = e^{(\mu-\gamma)y} N\left(\frac{y - \gamma t}{\sqrt{t}}\right) + e^{(\mu+\gamma)y} N\left(\frac{y + \gamma t}{\sqrt{t}}\right), \tag{58}$$

where $\gamma = \sqrt{2\lambda + \mu^2}$.

Recall the definition of the stopping time τ in Equation (23), which simplifies in the case with constant disagreement,

$$\tau = \inf\left\{t \geq 0 : D_t = \alpha S_0 e^{-\left(\mu - \sigma^2\right)(T-t)}\right\}. \tag{59}$$

Through a change of variables, we can redefine τ as the first hitting time of a drifted Brownian motion for a constant threshold. Specifically, define

$$y_t = \frac{1}{\sigma}\log\left(e^{-\left(\mu-\sigma^2\right)t} D_t\right), \tag{60}$$

then $y_0 = 0$,

$$y_t = Z_t^{\frac{\sigma}{2}} = \frac{\sigma}{2}t + Z_t. \tag{61}$$

Moreover,

$$\mathcal{T}_d^{\frac{\sigma}{2}} = \inf\{t \geq 0 : y_t = d\} \overset{a.s.}{=} \tau, \tag{62}$$

where the threshold is constant over time,

$$d = \frac{1}{\sigma} \log \left(\alpha S_0 e^{-\left(\mu - \sigma^2\right) T} \right). \tag{63}$$

Conditional on y_t and the fact that the circuit breaker has not been triggered up to time t, the result from (57) implies

$$P_t \left(\tau \leq T \right) = P_t \left(\mathcal{T}_{d_t}^{\frac{\sigma}{2}} \leq T - t \right) = N \left(\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}} \right) + e^{\sigma d_t} N \left(\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}} \right), \tag{64}$$

where

$$d_t = d - y_t = \frac{1}{\sigma} \left(\log \left(\frac{\alpha S_0}{D_t} \right) - \left(\mu - \sigma^2 \right) \left(T - t \right) \right). \tag{65}$$

The threshold d_t is normalized with respect to y_t so as to start the drifted Brownian motion $Z^{\frac{\sigma^2}{2}}$ from 0 at time t.

Next,

$$\begin{aligned} & \mathbb{E}_t \left[D_T^{-1} \mathbb{1}_{\{\tau > T\}} \right] \\ & = D_t^{-1} e^{-\left(\mu - \frac{\sigma^2}{2}\right)(T-t)} \mathbb{E}_t \left[e^{-\sigma (Z_T - Z_t)} \mathbb{1}_{\{\tau > T\}} \right] \\ & = D_t^{-1} e^{-\left(\mu - \sigma^2\right)(T-t)} \mathbb{E}_t \left[e^{-\sigma (Z_T - Z_t) - \frac{\sigma^2}{2}(T-t)} \mathbb{1}_{\{\tau > T\}} \right] \\ & = D_t^{-1} e^{-\left(\mu - \sigma^2\right)(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[\mathbb{1}_{\{\tau > T\}} \right] \\ & = D_t^{-1} e^{-\left(\mu - \sigma^2\right)(T-t)} \left[N \left(- \frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}} \right) - e^{-\sigma d_t} N \left(\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}} \right) \right]. \end{aligned} \tag{66}$$

The third equality follows from Girsanov's Theorem, and the fourth equality again follows from (57). Under $\mathbb{Q}$, $Z_t^{\sigma} = Z_t + \sigma t$ is a standard Brownian motion, and

$$y_t = -\frac{\sigma}{2} t + Z_t^{\sigma}. \tag{67}$$

Next, it follows from (60) and the definition of τ that

$$y_\tau = y_t + \frac{\sigma}{2} \left(\tau - t \right) + \left(Z_\tau - Z_t \right) = d. \tag{68}$$

Thus,

$$Z_\tau - Z_t = d_t - \frac{\sigma}{2}(\tau - t). \tag{69}$$

With this result, we can evaluate the following expectation,

$$\begin{aligned}
\mathbb{E}_t\left[\eta_\tau \mathbb{1}_{\{\tau\le T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_\tau - Z_t) - \frac{\delta^2}{2\sigma^2}(\tau - t)} \mathbb{1}_{\{\tau\le T\}}\right] \\
&= \eta_t e^{\frac{\delta d_t}{\sigma}} \mathbb{E}_t\left[\exp\left(-\left(\frac{\delta}{2} + \frac{\delta^2}{2\sigma^2}\right)(\tau - t)\right)\mathbb{1}_{\{\tau\le T\}}\right] \\
&= \eta_t \left(N\left(\frac{d_t - \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right) + e^{\left(\sigma + \frac{2\delta}{\sigma}\right)d_t} N\left(\frac{d_t + \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right)\right),
\end{aligned}$$

where the last equality follows from an application of (58).

Finally,

$$\begin{aligned}
\mathbb{E}_t\left[\eta_T D_T^{-1} \mathbb{1}_{\{\tau > T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_T - Z_t) - \frac{\delta^2}{2\sigma^2}(T-t)} D_t^{-1} e^{-\left(\mu - \frac{\sigma^2}{2}\right)(T-t) - \sigma(Z_T - Z_t)} \mathbb{1}_{\{\tau > T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu - \sigma^2 + \delta\right)(T-t)} \mathbb{E}_t\left[e^{\left(\frac{\delta}{\sigma} - \sigma\right)(Z_T - Z_t) - \frac{\left(\frac{\delta}{\sigma} - \sigma\right)^2}{2}(T-t)} \mathbb{1}_{\{\tau > T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu - \sigma^2 + \delta\right)(T-t)} \mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[\mathbb{1}_{\{\tau > T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu - \sigma^2 + \delta\right)(T-t)} \left(N\left(-\frac{d_t - \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right)\right. \\
&\quad \left. - e^{\left(\frac{2\delta}{\sigma} - \sigma\right)d_t} N\left(\frac{d_t + \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right)\right).
\end{aligned}$$

The third equality follows from Girsanov's Theorem, and the fourth equality follows from (57). Under $\widetilde{\mathbb{Q}}$, $Z_t^{\sigma - \frac{\delta}{\sigma}} = Z_t + \left(\sigma - \frac{\delta}{\sigma}\right)t$ is a standard Brownian motion, and

$$y_t = \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)t + Z_t^{\sigma - \frac{\delta}{\sigma}}. \tag{70}$$

□

A.4 Proof of Proposition 3

Sketch of the proof: Consider the function $f(s)$ as the time 0 stock price in an economy where the stock market closes the first time when the stock price reaches some exogenous threshold s. When $s = 0$, there are effectively no circuit breakers. Thus,

$$f(0) = \widehat{S}_0 \geq \min\left(D_0 e^{(\mu-\sigma^2)T}, D_0 e^{\left(\mu-\sigma^2\right)T - a(0,T) - b(0,T)\delta_0}\right) \equiv \bar{s}.$$

The inequality follows from the fact that $\widehat{S}_0$ is the weighted harmonic average of the prices of the stock in two single-agent economies. Next, it can be shown that $f(s)$ is a continuous function and moreover is decreasing in s. The intuition is that as s increases, the circuit breaker constraint becomes more binding, making it more likely for the stock price to converge to the lower valuation between the two agents. Finally, when s rises to $\bar{s}$, the stock market will close immediately, which means $f(\bar{s}) = \bar{s}$. Now it is easy to see that $f(s)$ and $\frac{s}{\alpha}$ will have a single crossing in the region $(0, \bar{s})$ for any $\alpha \in [0, 1]$.

B Constant Disagreements: $\delta > 0$

Analogous to Figure 2 - Figure 5, Figure 13 - Figure 16 present the set of results for the case of constant disagreement with $\delta = 2\%$.

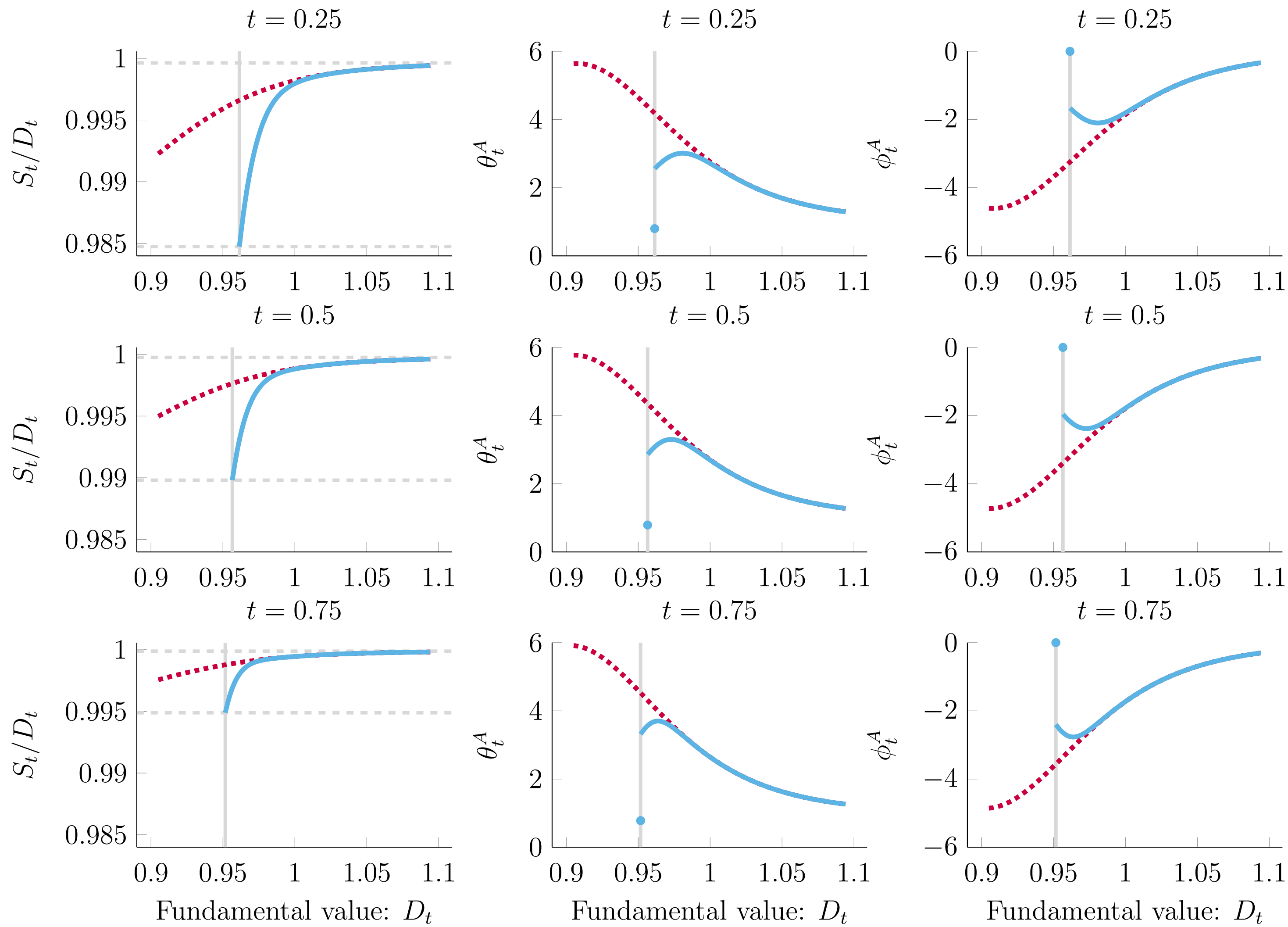

Figure 2: **Price-dividend ratio and agent A's (rational optimist) portfolio holdings.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

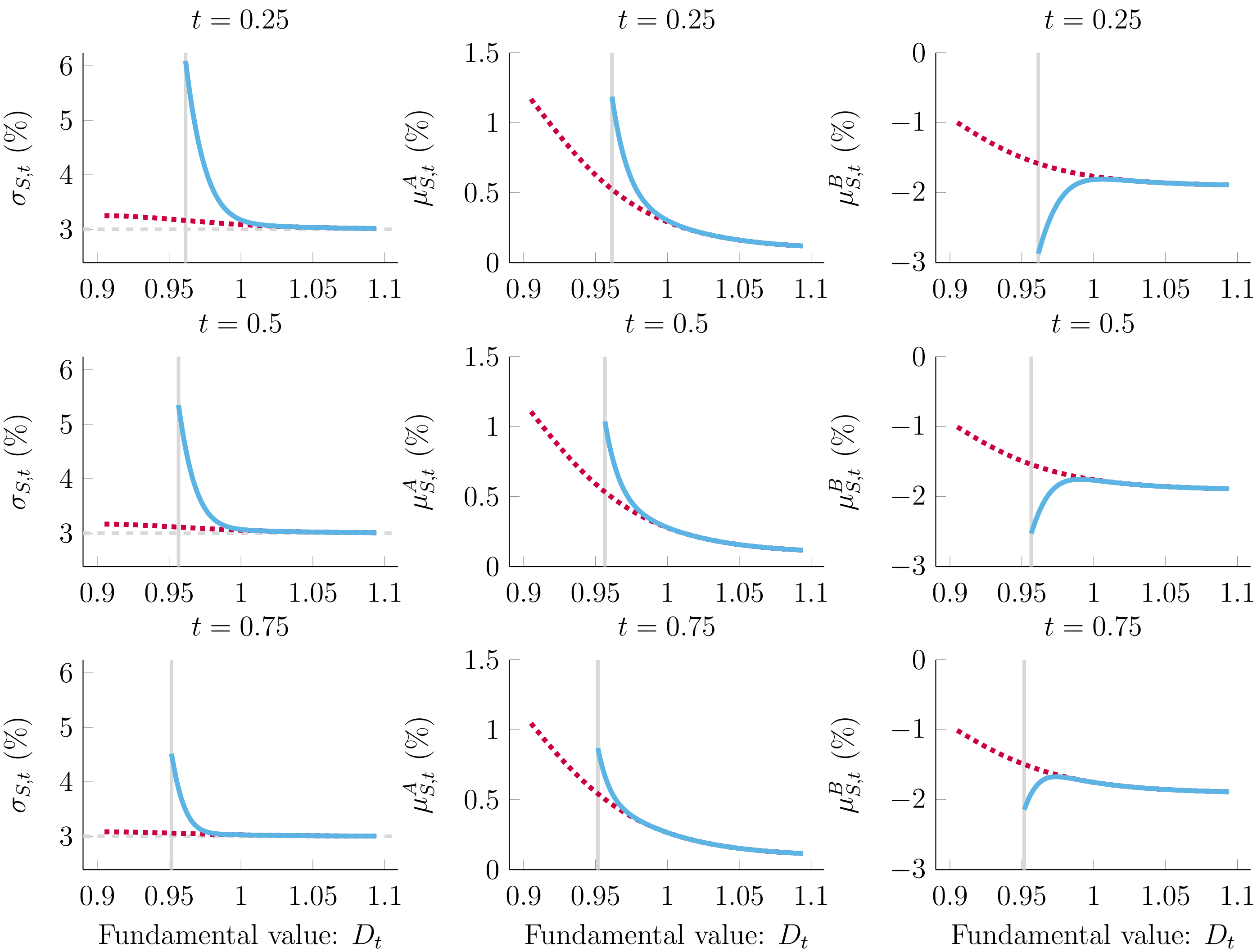

Figure 3: **Conditional volatility and conditional expected returns under the two agents' beliefs.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

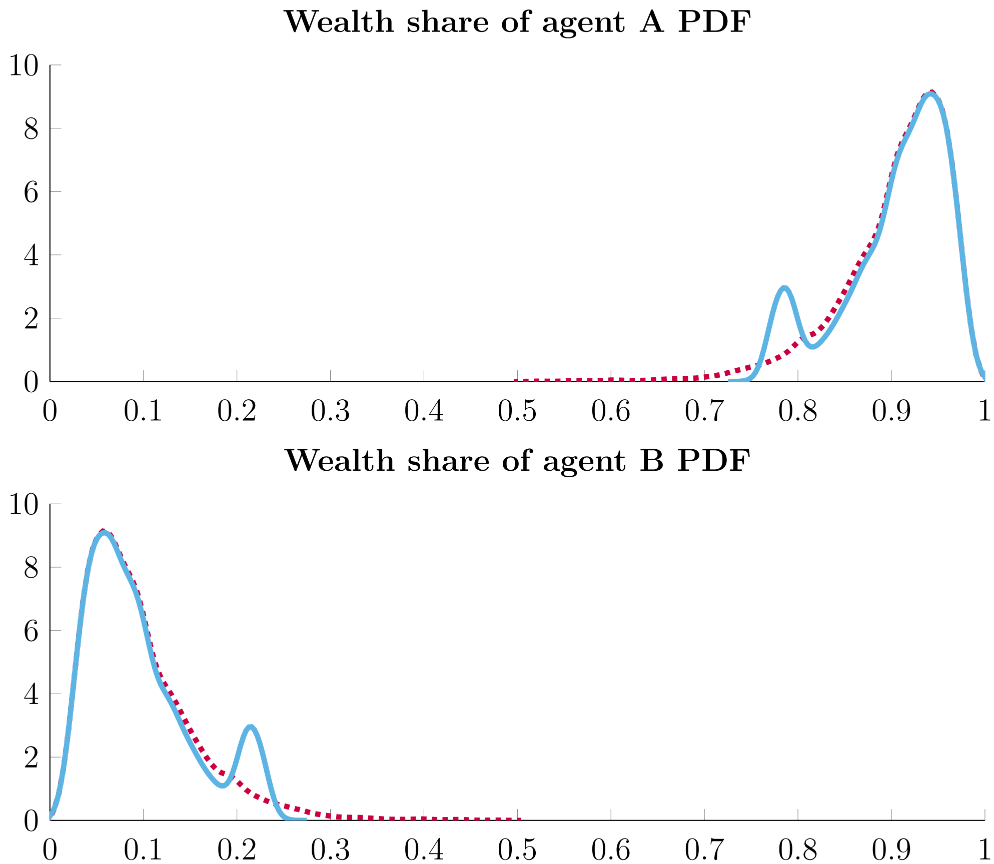

Figure 4: **Distribution of terminal wealth share.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

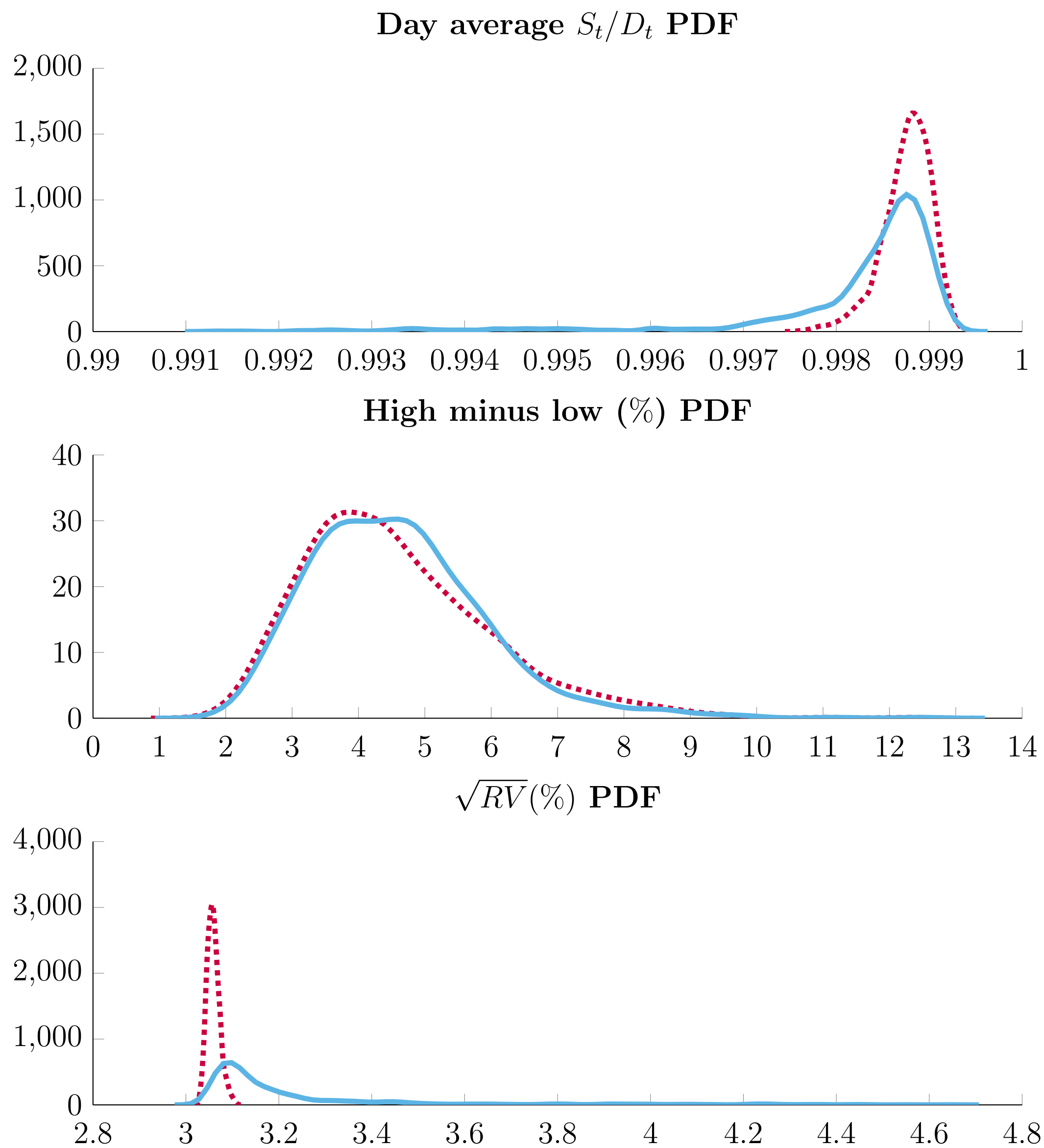

Figure 5: **Distributions of price-dividend ratio, daily price range, and realized volatility.** Solid line: circuit breaker is on; dashed line: complete markets.

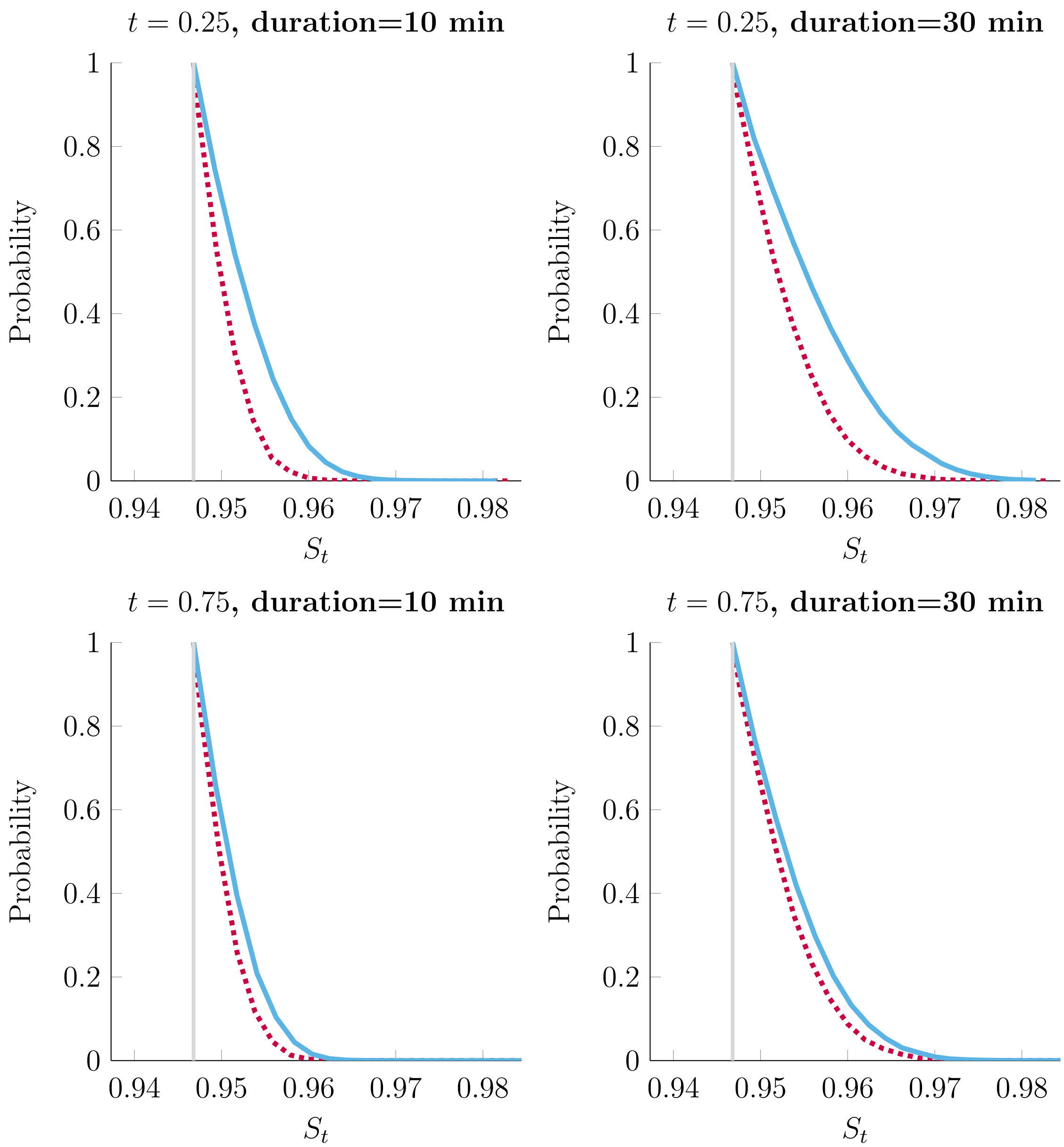

Figure 6: **The "magnet effect".** Probabilities for the stock price to reach the circuit breaker limit within 10 and 30 minutes.

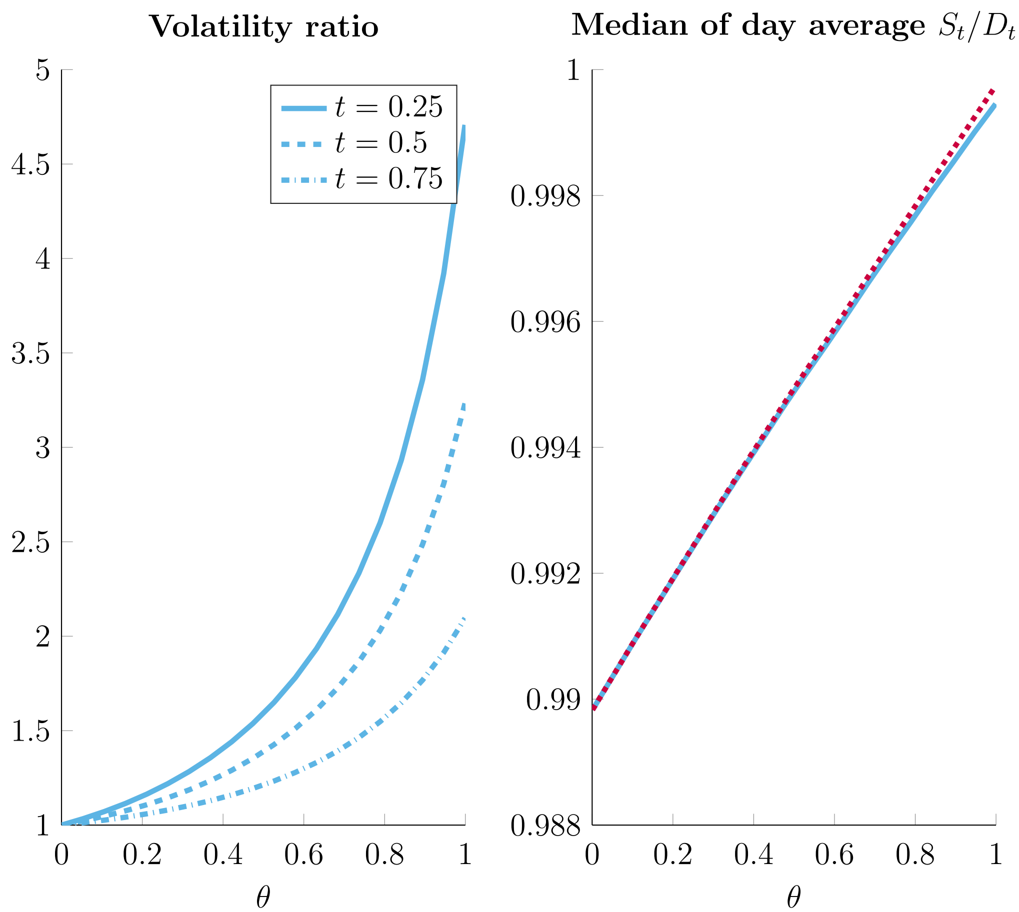

Figure 7: **Comparative statics for the initial wealth distribution.** The left panel plots the ratio of conditional return volatilities for the with-CB and no-CB cases at $D_t = \underline{D}(t)$ as a function of θ, the initial wealth share for the rational agent A. The right panel plots the median of daily average price-dividend ratio as a function of θ.

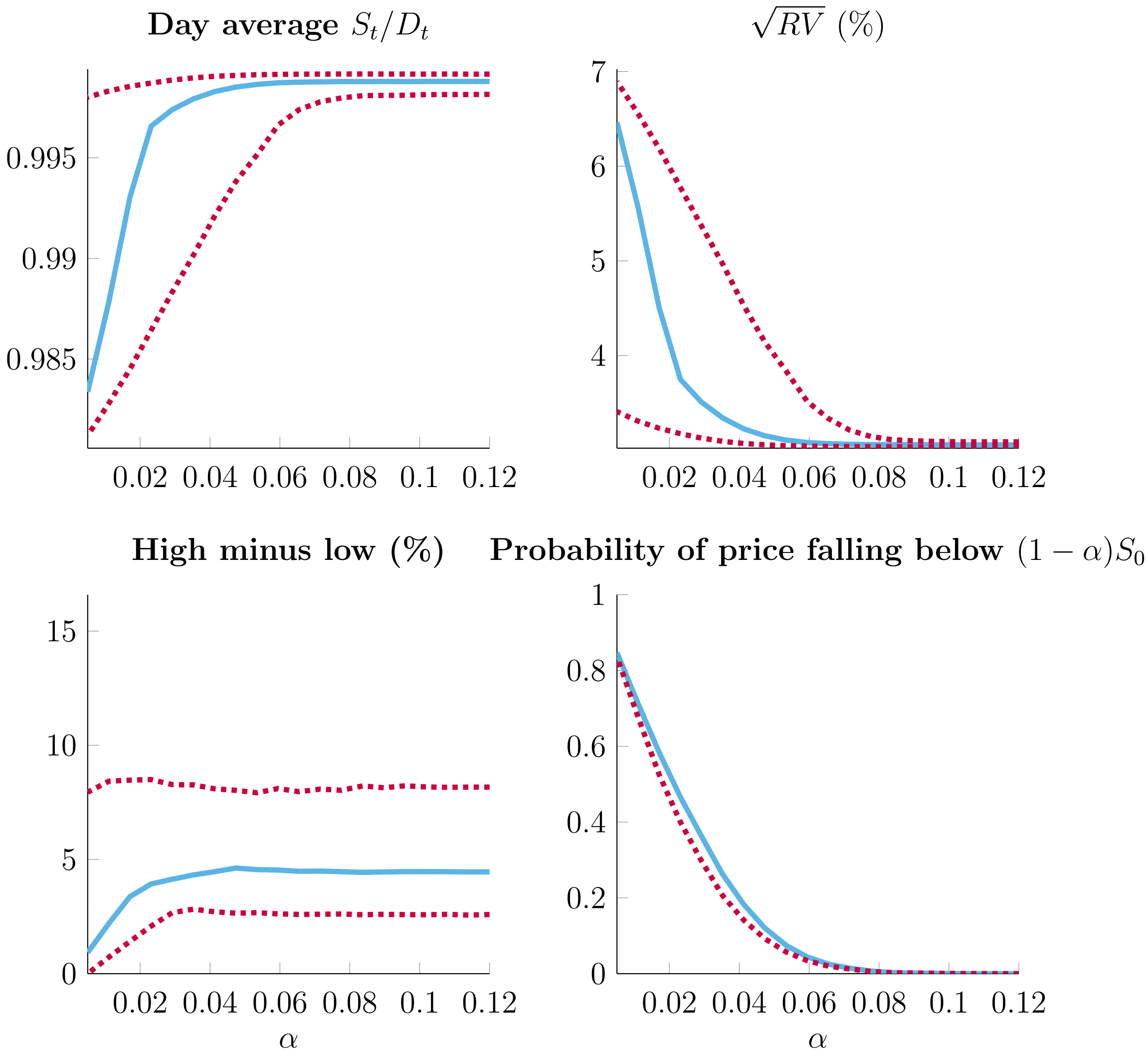

Figure 8: **Comparative statics for the circuit breaker limit α.** This figure plots the daily average price-dividend ratio, the daily realized volatility, daily price range, and daily probability of reaching the circuit breaker threshold as a function of the circuit breaker limit α. In Panels A-C, the dash lines denote 95% percentiles, and the solid lines denote the median values across simulations. In Panel D, the dash line denotes complete markets, while the solid line denotes the circuit breaker case.

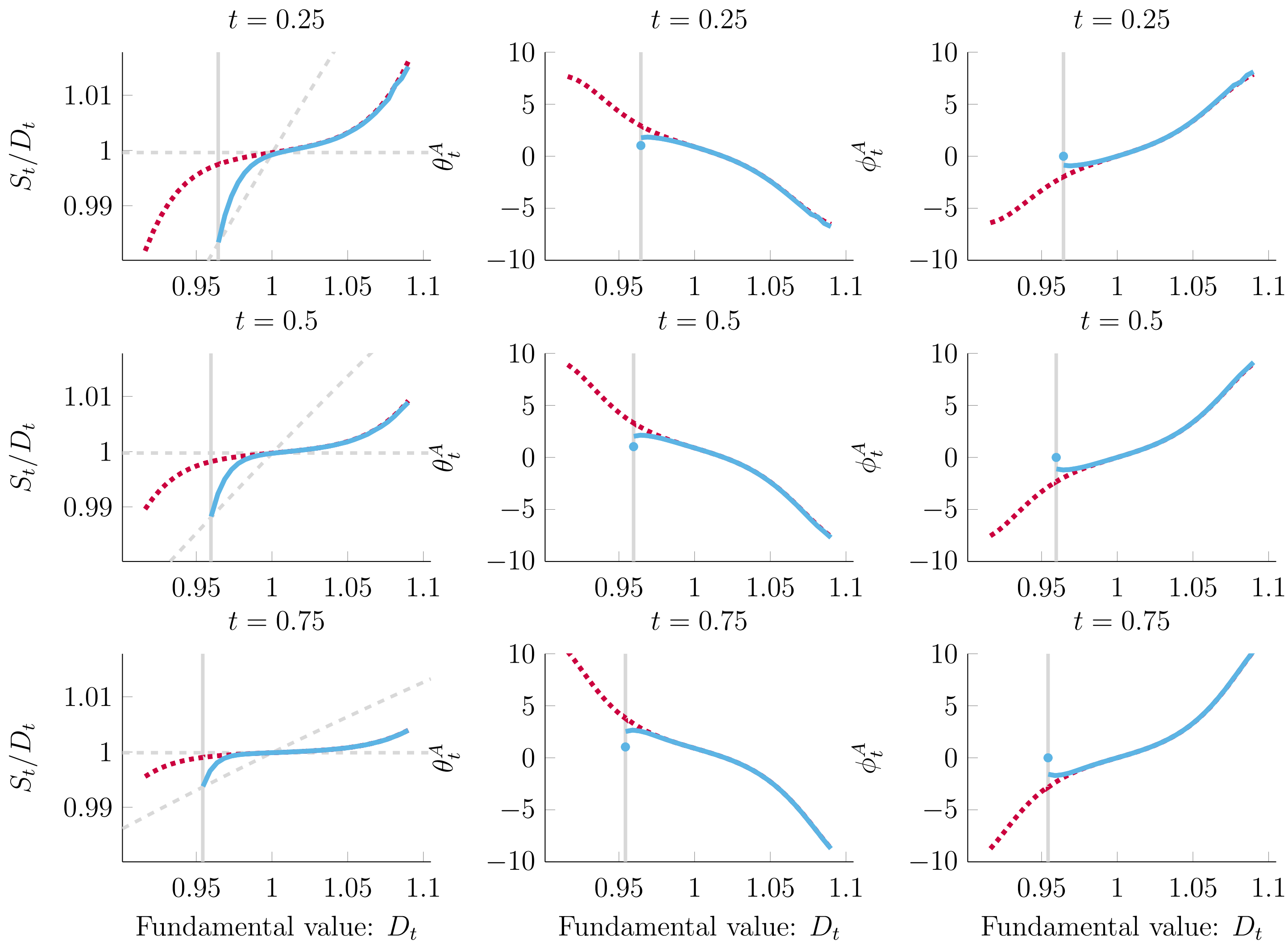

Figure 9: **Price-dividend ratio and agent A's portfolio in the case of time-varying disagreements.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

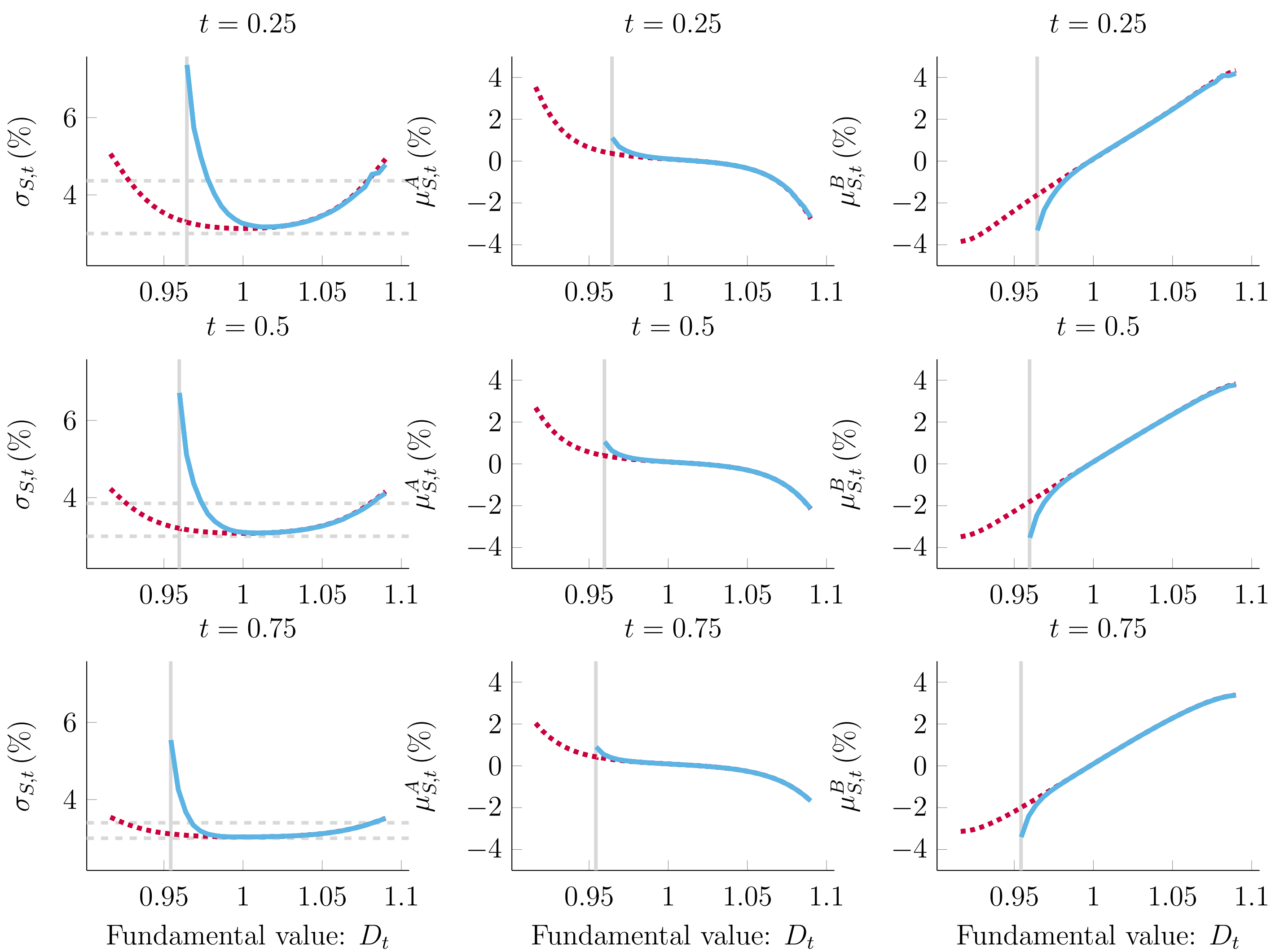

Figure 10: **Conditional volatility and conditional expected returns in the case of time-varying disagreements.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

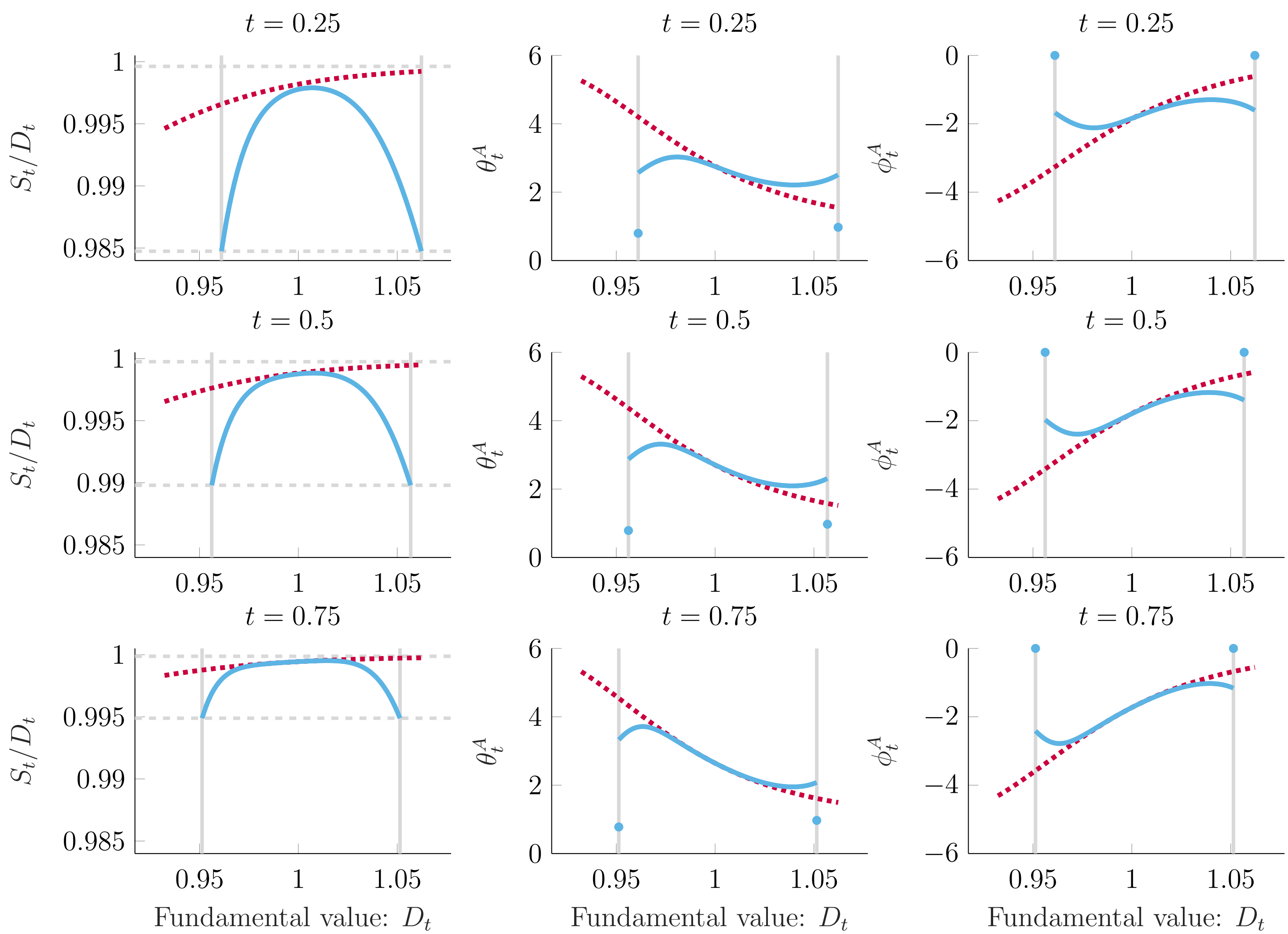

Figure 11: **Price-dividend ratio and agent A's (rational optimist) portfolio holdings.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

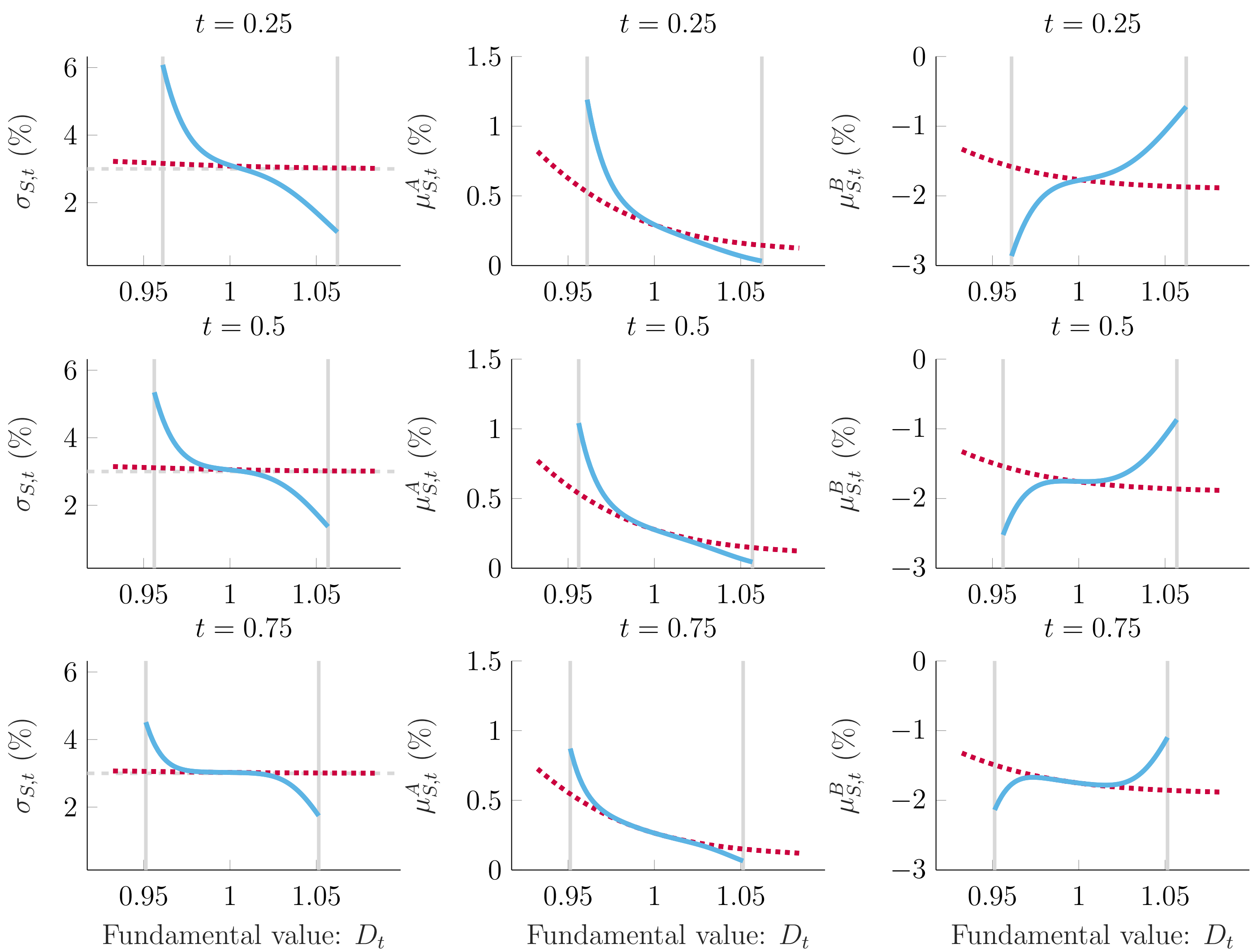

Figure 12: **Conditional volatility and conditional expected returns under the two agents' beliefs.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

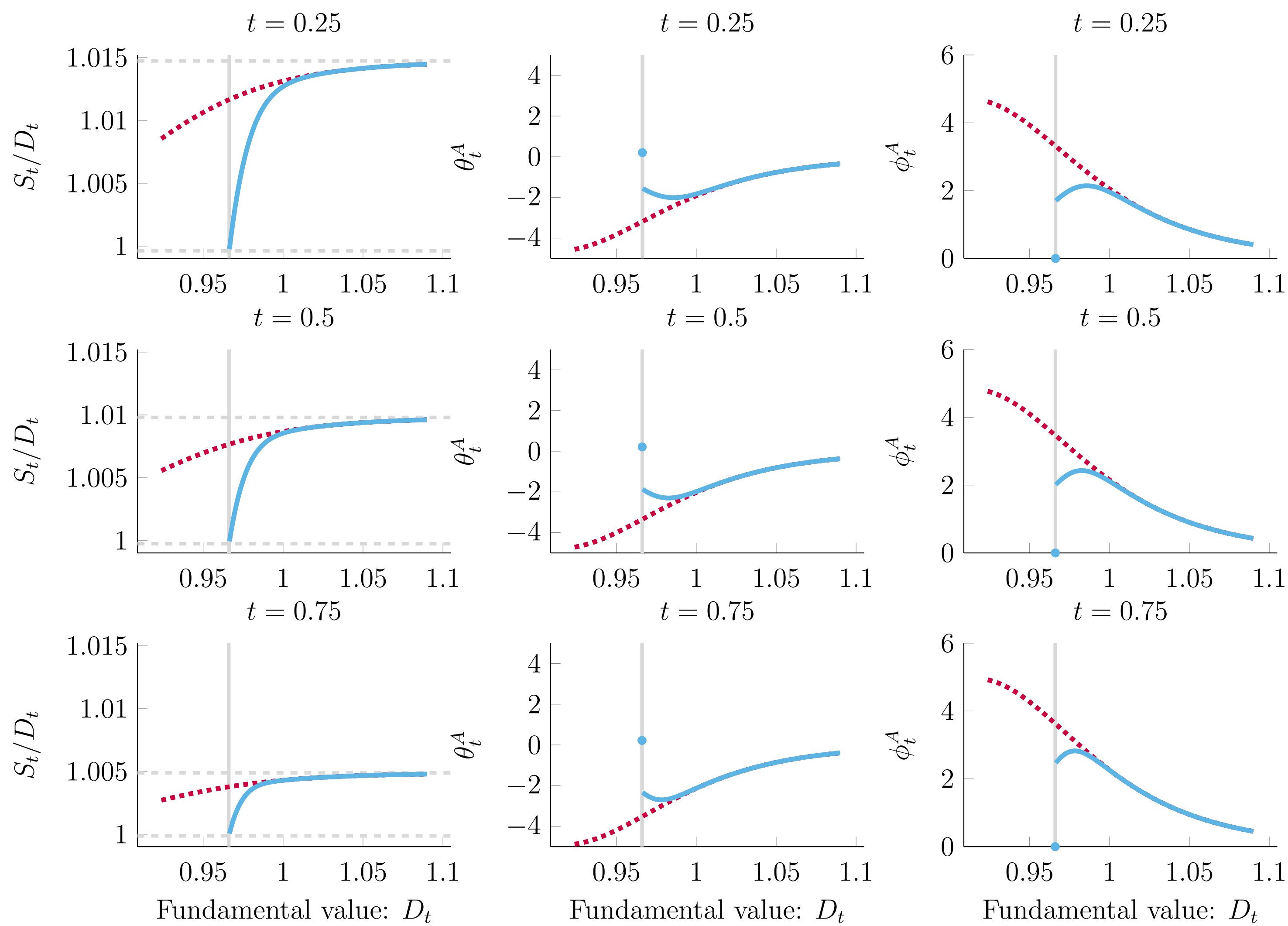

Figure 13: **Price-dividend ratio and agent A's (rational pessimist) portfolio holdings.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

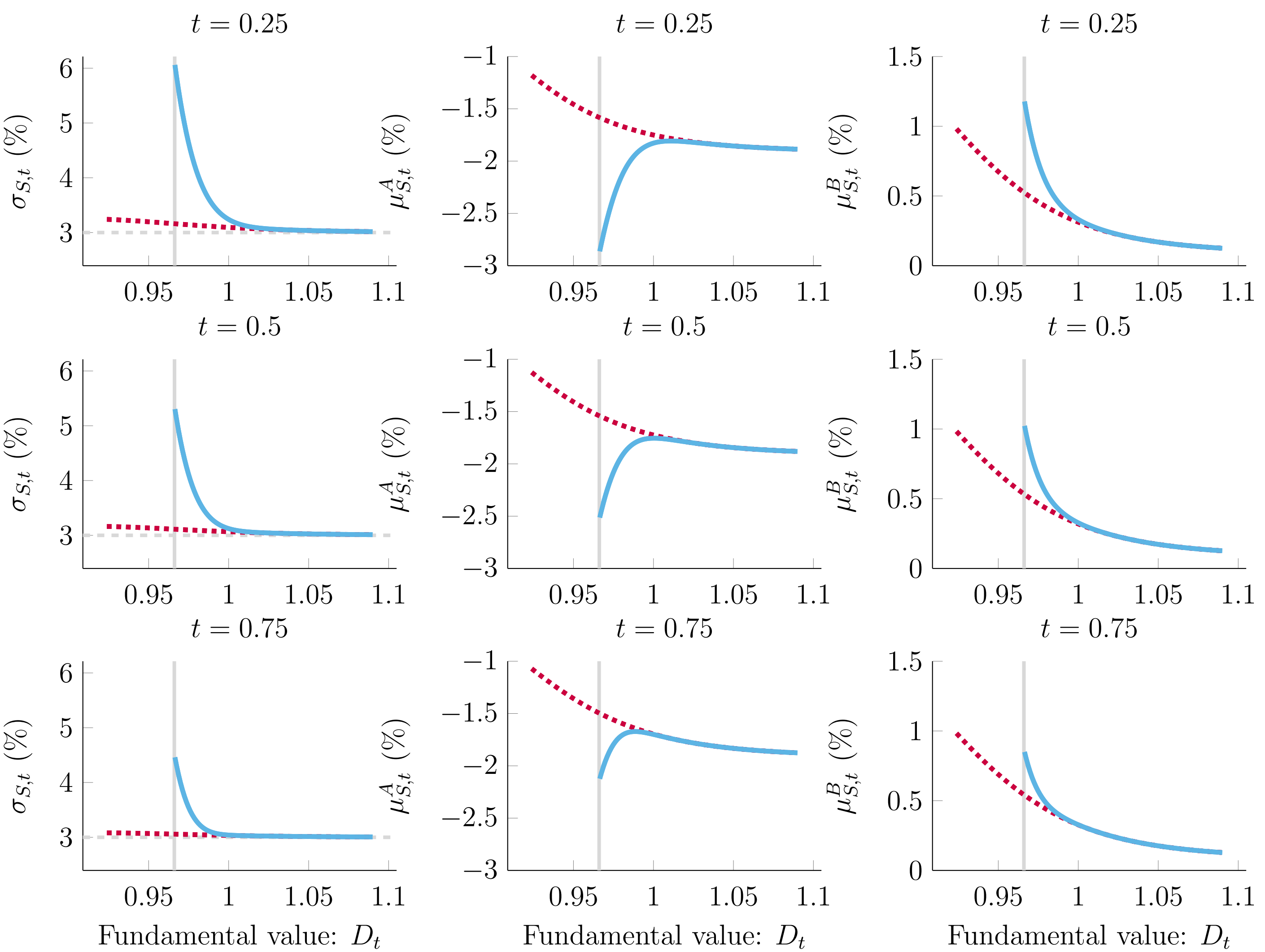

Figure 14: **Conditional volatility and conditional expected returns under the two agents' beliefs.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

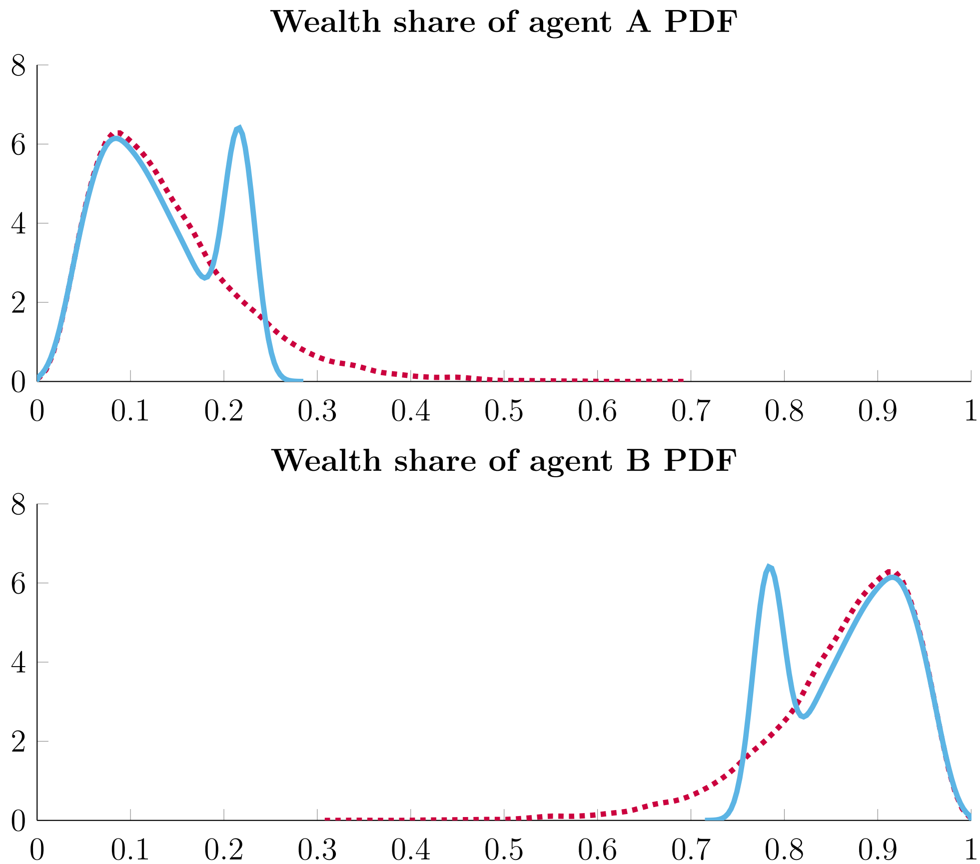

Figure 15: **Distribution of terminal wealth share.** Blue solid lines are for the case with circuit breakers. Red dotted lines are for the case without circuit breakers. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

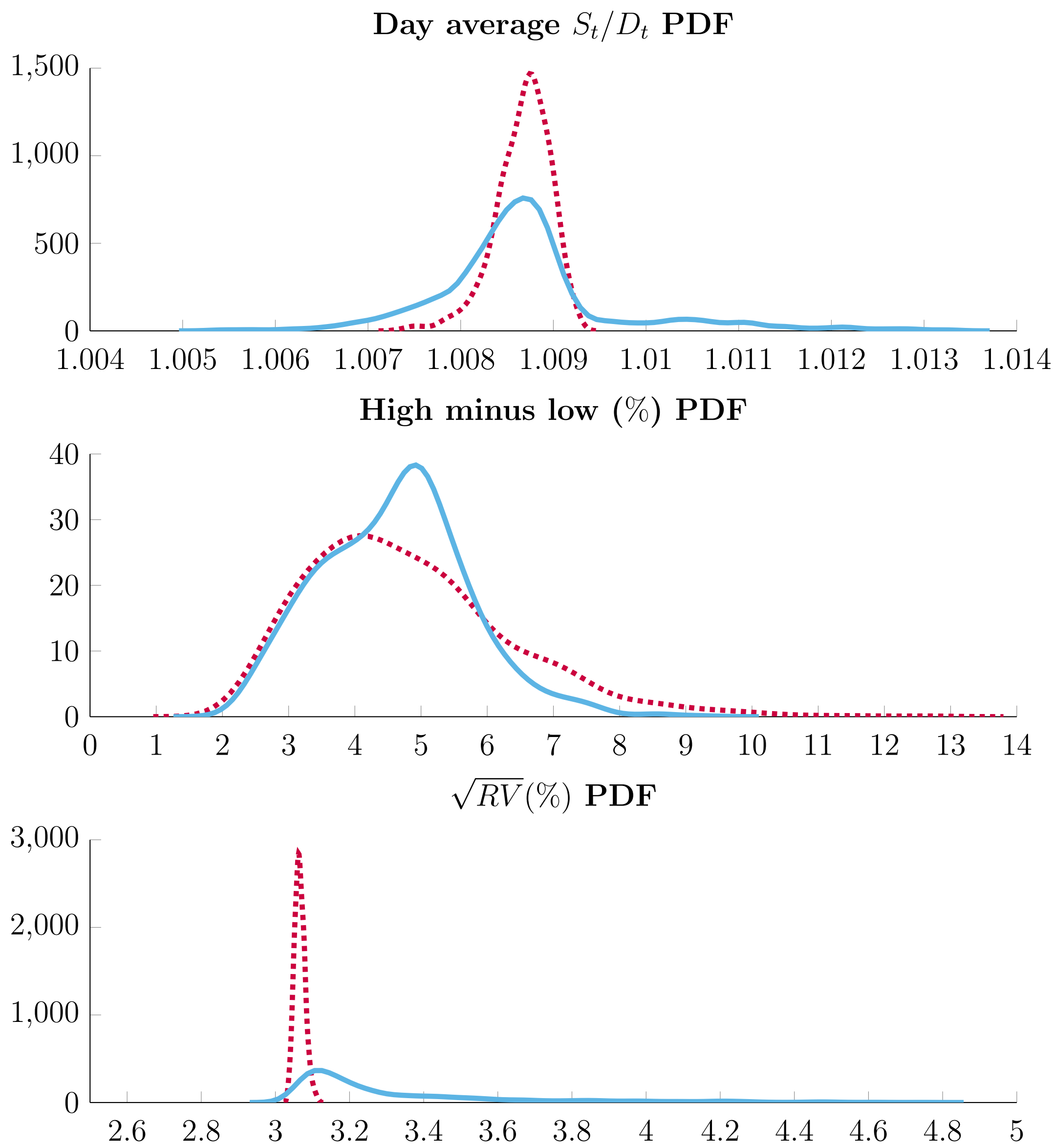

Figure 16: **Distributions of price-dividend ratio, daily price range, and realized volatility.** Solid line: circuit breaker is on; dashed line: complete markets.

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